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Technical Report

A Semantics for Reductive Logic and Proof-search

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A Semantics for Reductive Logic and Proof-search

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Abstract

Since its earliest presentations, mathematical logic has been formulated as a formalization of *deductive* reasoning: given a collection of hypotheses, a conclusion is derived. However, the advent of computational logic has emphasized the significance of *reductive* reasoning: given a putative conclusion, what are sufficient premisses? Whilst deductive systems typically have a well-developed semantics of proofs, reductive systems are typically well-understood only operationally. Typically, a deductive system can be read as a corresponding reductive system. The process of calculating a proof of a given putative conclusion, for which non-deterministic choices between premisses must be resolved, is called *proof-search* and is an essential enabling technology throughout the computational sciences. We suggest that the reductive view of logic is (at least) as fundamental as the deductive view and discuss some of the problems which must be addressed in order to provide a semantics of reductions and searches of comparable value to the corresponding semantics of proofs. Just as the semantics of proofs is intimately related to the model theory of the underlying logic, so too should be the semantics of reduction and of proof-search. We discuss how to solve the problem of providing a semantics for proof-searches which adequately models both not only the logical but also the operational aspects of the reductive system.

1 Introduction

Axiomatizations of logics as formal systems are usually formulated as calculi for deductive inference. Deductive inference proceeds from established or supposed premisses to a conclusion, regulated by the application of *inference rules*, R ,

$$\Downarrow \frac{\text{Premiss}_1 \dots \text{Premiss}_m}{\text{Conclusion}} R.$$

A proof is constructed, inductively, by applying instances of rules of this form to proofs of established premisses, thereby constructing a proof of the given conclusion.

A conceptually valuable semantics of proofs is provided by a correspondence between the propositions and proofs of a logic, the types and terms of a λ -calculus and the objects and arrows of a category *q.v.* Figure 1, in which (*e.g.*, natural deduction) proofs correspond to (*e.g.*, typed λ -terms) which correspond to classes arrows in categories with specified structure.

Reductive inference proceeds from a putative conclusion to sufficient premisses, regulated by *reduction operators*, O_R ,

$$\Uparrow \frac{\text{Sufficient Premiss}_1 \dots \text{Sufficient Premiss}_m}{\text{Putative Conclusion}} O_R,$$

corresponding to (admissible) rules, R .¹ We believe that this idea of reduction was first explained *in these terms* by Kleene [17]. A reduction is constructed, inductively, by applying instances of reduction operators of this form to putative conclusions of which a proof is desired, thereby yielding

¹Henceforth we refer to just R rather than O_R .

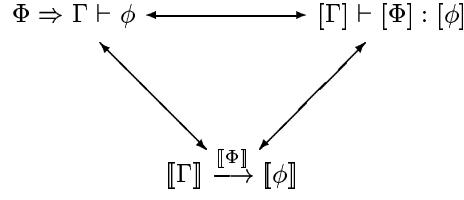


Figure 1: Propositions-as-types-as-objects

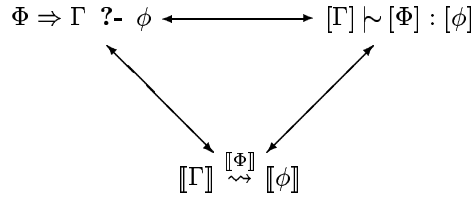


Figure 2: Reductions-as-realizers-as-arrows

a collection of sufficient premisses, proofs of which would be sufficient to imply the existence of a proof, obtainable by deduction, of the putative conclusion.

Starting from a given endsequent,² we apply reduction operators until either we have constructed a proof or we determine that our reduction cannot be further extended to yield a proof. Such a determination will certainly occur if, on at least one branch of our reduction, we have reduced as far as a putative conclusion of the form

$$p_1, \dots, p_m \text{ ?- } q,$$

read as querying whether q is a consequence of p_1, \dots, p_m , and in which all of the p_i s and the q are atomic and no p_i is identical to q .³ In this situation, we have not terminated the branch by reaching an axiom and no further reduction will help.

This inherent partiality of reductions presents a clear semantic challenge: we must be able to interpret those not only proofs, which are a special class of reductions, but also reductions which cannot be completed to be proofs. Nevertheless, we aim to recover a semantics for reductions of utility comparable to that of the propositions-as-types-as-objects triangle for proofs.

The desired set-up is summarized in Figure 2, in which $\Gamma \text{ ?- } \phi$ denotes a sequent which is a putative conclusion and $\Phi \Rightarrow \Gamma \text{ ?- } \phi$ denotes that Φ is a reduction with root $\Gamma \text{ ?- } \phi$. The judgement $[\Gamma] \sim [\Phi] : [\phi]$ indicates that $[\Phi]$ is a *realizer* of $[\phi]$ with respect to assumptions $[\Gamma]$.

The provision of such a framework, which we require to be adequate model-theoretically, is non-trivial. The main difficulty is that the objects constructed during a reduction, are, in contrast to the objects, *i.e.*, proofs, constructed during deduction, inherently *partial*. Whilst any deduction proceeds from axioms to a guaranteed conclusion and so constructs a proof, reductions proceed from a putative conclusion to sufficient premisses. At any intermediate stage, it can be that it is impossible to complete the reduction so as to obtain a proof, *i.e.*, all possible reductions lead to

²That is, a sequent occurring at the root of a proof/reduction tree.

³We restrict our attention to single-conclusioned sequents in this discussion just for simplicity.

trees in which there are leaves of the form $\phi \text{ ?- } \psi$ in which the formulæ ϕ and ψ are both distinct and irreducible.⁴

Suppose, then, that we have a deductive system D which is interpreted in a category \mathcal{C} . Consider the interpretation of an axiom sequent, $\phi \vdash \phi$, given by

$$\llbracket \phi \rrbracket \xrightarrow{1_{\llbracket \phi \rrbracket}} \llbracket \phi \rrbracket,$$

the identity arrow from $\llbracket \phi \rrbracket$ to itself. Proof trees over D have the property that all leaves have this form (or something very like it).

Now consider the reductive system $R(D)$, obtained by reading each of D 's inference rules as reduction operators. Reduction trees over $R(D)$ can have leaves of the form $\phi \vdash \psi$, where ϕ and ψ are distinct, irreducible formulæ, so that there is no way to reduce the leaf to an axiom of the deductive system. A semantics of reductions in $R(D)$ must interpret leaves of this form. One solution is to interpret reductions not in the category \mathcal{C} but in the polynomial category $\mathcal{C}[\alpha]$ over an indeterminate α .⁵

[**Aside:** If A and B are objects of a category \mathcal{C} , we can adjoin an indeterminate $A \xrightarrow{\alpha} B$ by forming the *polynomial category* $\mathcal{C}[\alpha]$. The objects of $\mathcal{C}[\alpha]$ are the objects of \mathcal{C} and the arrows of $\mathcal{C}[\alpha]$ are formed freely from the arrows of \mathcal{C} together with the new arrow α . The basic ideas may be found in [20].]

Then the interpretation of a leaf of the form $p \text{ ?- } q$, where p and q denote propositional letters,⁶ can be defined as follows:

$$\llbracket p \rrbracket \xrightarrow{\alpha} \llbracket q \rrbracket.$$

The corresponding language of realizers is the internal language of $\mathcal{C}[\alpha]$.

Whilst polynomials over categories of proofs provide a place within which reductions can be interpreted, there is much more to consider in the semantics of *proof-search*.

So far we have discussed just reduction as a (declarative) counterpart to deduction. But the *construction* of a reduction requires an algorithmic control régime. For example, when faced with an endsequent $\Gamma \text{ ?- } \phi$ we must choose which reduction operator to apply first and to which formula. This is a disjunctive choice: we need only find one choice of operator and formula for which the successful construction of a proof is possible.

Having chosen and applied an operator, we obtain, in general, several sufficient premisses,

$$\Gamma_1 \text{ ?- } \phi_1, \dots, \Gamma_m \text{ ?- } \phi_m.$$

We must then choose which sequent to reduce first. In order to successfully construct a proof, all of the sequents must be developed — this is a conjunctive choice — but some choices might lead quickly to failure, while others might fail only after a great deal of computation.

Upon failure, we must return to a point at which a disjunctive choice was made. That is, we must *backtrack*. Whilst a complete description of the control régime for constructing a proof would also require the modelling of the non-deterministic choices described above, the key control mechanism is backtracking. Thus our semantics of proof-search is focussed on the incorporation of backtracking into our model-theoretic semantics for reductive logic.

The key point is that the denotational semantics of the search process is inherently *intuitionistic*: the search procedure, we might think of it as an *agent*, can be seen as van Dalen's *creative subject* [38] exploring a Kripke frame (W, \sqsubseteq) in which the ordering is generated by the reduction

⁴We say that an occurrence of a formula ϕ , in a reduction tree over a system S , is *irreducible* if it is not the principal formula of an instance of any reduction operator of S .

⁵In general, the polynomial over a set of indeterminates.

⁶We call such a leaf *atomic*.

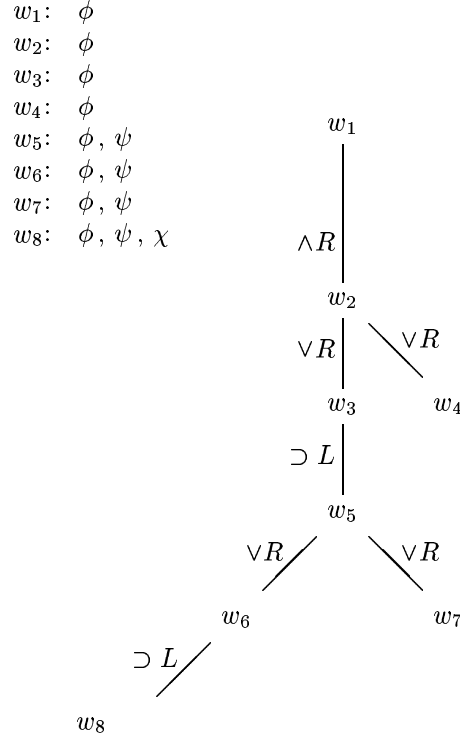


Figure 3: Reduction Semantics as Kripke Semantics

operators of the logic. At each reduction, starting from a given endsequent, the creative subject increases his knowledge of established atomic propositions. For example, illustrated in Figure 3, suppose we have, in intuitionistic logic, the endsequent

$$\phi; \phi \supset \psi; \phi \supset \chi \vdash (\psi \vee \psi') \wedge (\chi \vee \chi'),$$

in which ϕ, ψ, ψ', χ and χ' are atomic. At the root world, w_1 , the only atomic proposition established on the left, and so potentially capable, in the presence of a matching ϕ on the right, of forming an axiom sequent $\phi \vdash \phi$, is ϕ .⁷

The next two reductions, $\wedge R$ and $\vee R$, take us to worlds w_2 and then w_3 and w_4 without adding to the atomic propositions established on the left. Next comes a $\supset L$, with principal formula $\phi \supset \psi$. This step adds ψ to the atomic formulæ established on the left, and so capable of contributing to axioms. As before, the accession to worlds w_6 and w_7 , via an $\vee R$, adds no atoms to the left. Finally, the $\supset L$ leading to w_8 adds χ to the collection of formulæ established on the left.

Thus, beginning with polynomials in § 2, we take, in § 2.1, Kripke-like structures as the basis for our semantics of reductive logic. We give a game-theoretic example in § 2.2. In § 3, we explain the representation of classical logic that is provided by the $\lambda\mu\nu$ -calculus, and introduce its (fibred) models. In § 4, this semantics is used to provide, for intuitionistic logic, a basis for incorporating a semantics of proof-search via an embedding in a classical structure in which the additional ‘states’ are used to represent backtracking. We provide two concrete examples of such a semantics: In § 5, using continuations, and in § 6, using games.

⁷Our use of just atoms to form axioms should be considered analogous to the use of atoms in a least Herbrand model [39].

$\frac{}{\Gamma, \phi \vdash \phi} Ax$	$\frac{\Gamma, \psi \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi} Cut$
$\frac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \psi, \phi \vdash \Delta} ExchangeL$	
$\frac{}{\Gamma, \perp \vdash \phi} \perp L$	$\frac{}{\Gamma \vdash \top} \top R$
$\frac{\Gamma, \psi, \psi' \vdash \phi}{\Gamma, \psi \wedge \psi' \vdash \phi} \wedge L$	$\frac{\Gamma \vdash \phi \quad \Gamma \vdash \phi'}{\Gamma \vdash \phi \wedge \phi'} \wedge R$
$\frac{\Gamma \vdash \phi \quad \Gamma, \psi \vdash \chi}{\Gamma, \phi \supset \psi \vdash \chi} \supset L$	$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \supset \psi} \supset R$
$\frac{\Gamma, \phi \vdash \psi \quad \Gamma, \phi' \vdash \psi}{\Gamma, \phi \vee \phi' \vdash \psi} \vee L$	$\frac{\Gamma \vdash \phi \quad \Gamma \vdash \phi'}{\Gamma \vdash \phi \vee \phi'} \vee R$

Table 1: Intuitionistic Sequent Calculus

2 Semantics for Intuitionistic Reductive Logic

In this section, we describe a semantics for propositional intuitionistic logic viewed as a reductive system. Building on the wealth of proof-theoretic studies of proof-search in intuitionistic logic [7, 33, 31, 30, 28, 40], we take as our point of departure a minor variant of Gentzen's sequent calculus, LJ' given in Table 1, in which Contraction and Weakening are built into the other rules. However, for technical reasons, we include, and emphasize, *ExchangeL*. For convenience, we shall simply refer to this system as LJ.

The principal virtues of LJ's presentation of intuitionistic proofs as a basis for mechanical proof-search are that it admits Cut-elimination and, in contrast to natural deduction systems, has, in the absence of Cut, the *subformula property*. Note, that although Cut forms the basis of the resolution procedure used by Prolog [34, 19, 39], one can simulate the analytic cuts used in resolution by implication in LJ (see [31]) hence it is possible to use LJ also as a calculus to study analytic resolution. However, for proof-search either wholly or partially by humans, the Cut rule is very useful because it allows the use of lemmas in proofs and leads to shorter proofs [37, 3].

An LJ *reduction* is a tree regulated by the operators of LJ, *i.e.*, the inference rules of LJ read as reduction operators, from conclusion to premisses. As usual, the sequent $\Gamma \vdash \Delta$ at the root of a tree is called an *endsequent*. We use the following notations for reductions: We write $R_1; \dots; R_n$ for a reduction with operators $R_1; \dots; R_n$ applied in which the putative conclusion of every R_i , $i \geq 2$, is one of the sufficient premisses of some operator R_j , for $j < i$.

As we have explained in the introduction, a major difference between reductions and proofs is that reductions need not have axiom sequents at their leaves. Whereas all of the leaves of a proof are of the form $\Gamma, \phi, \Gamma' \vdash \phi$, reductions may have leaves of the form $p \vdash q$, where p and q are distinct propositional letters. Although a branch with such a leaf cannot be extended so as to obtain just axioms at its leaves, a semantics of must nevertheless reductions must give meaning to these reductions of this form.

In order to give a semantics for reductions, we start by reviewing our first main tool, namely polynomial categories. These polynomial categories are used to model partial reductions.

Definition 1 *Let \mathcal{C} be a bi-Cartesian closed category, and let A, B be two objects of \mathcal{C} . The polynomial category $\mathcal{C}(\xi)$ over an indeterminate $\xi : A \rightarrow B$ is the free bi-Cartesian closed category over the graph of \mathcal{C} with an additional edge ξ with source A and target B modulo the equations in \mathcal{C} .*

Polynomial categories have a universal property similar to polynomials over the natural numbers [20]:

Theorem 2 *Let \mathcal{C} be a bi-Cartesian closed category and $\mathcal{C}(\xi)$ be the polynomial category over the indeterminate $\xi: A \rightarrow B$.*

- (i) *For every bi-Cartesian closed functor $F: \mathcal{C} \rightarrow \mathcal{D}$ and any morphism $f: FA \rightarrow FB$ in \mathcal{D} , there is a bi-Cartesian closed functor $\hat{F}: \mathcal{C}(\xi) \rightarrow \mathcal{D}$.*
- (ii) *Any bi-Cartesian closed functor $G: \mathcal{C}(\xi) \rightarrow \mathcal{D}$ is equal to \hat{F} for some bi-Cartesian closed functor $F: \mathcal{C} \rightarrow \mathcal{D}$ and morphism $f: FA \rightarrow FB$ in \mathcal{D} such that $\hat{F}(\xi) = f$.*

Proof Direct consequence of the freeness of a polynomial category. \square

We will write $\mathcal{C}(\xi_1, \dots, \xi_n)$ for $(\dots((\mathcal{C}(\xi_1)(\xi_2) \dots)(\xi_n)) \dots)$. We call a functor

$$\hat{F}: \mathcal{C}(\xi) \rightarrow \mathcal{C}(\xi_1, \dots, \xi_n),$$

obtained by the universal property from the inclusion functor $\mathcal{C} \rightarrow \mathcal{C}(\xi_1, \dots, \xi_n)$ and a morphism f in $\mathcal{C}(\xi_1, \dots, \xi_n)$, a *substitution functor* and write $S_\xi(f)$ for such a functor. This functor is the analogue to substitution of natural numbers for indeterminates in polynomials over natural numbers.

The polynomial category can be defined in more standard categorical terms if the indeterminate ξ is a morphism $\xi: 1 \rightarrow A$, where the domain is the terminal object. Such a morphism is called a *global section*, and in the case of $\mathcal{C} = \text{Set}$ corresponds to an element of the set A . This restriction does not cause a loss of generality: an indeterminate $\xi: A \rightarrow B$ corresponds via the universal property defining function spaces to an indeterminate $\xi': 1 \rightarrow A \Rightarrow B$. The equivalent definition using standard terms is as follows:

Proposition 3 *Suppose \mathcal{C} is a bi-Cartesian closed category. Each polynomial category $\mathcal{C}(\xi)$ with an indeterminate $\xi: 1 \rightarrow A$ is isomorphic to the co-Kleisli category \mathcal{D} for the endofunctor $(- \times A)$ on \mathcal{C} .*

Proof Firstly, it is a routine check that the co-Kleisli category is bi-Cartesian closed and that the inclusion $\mathcal{C} \rightarrow \mathcal{D}$ is a bi-Cartesian closed functor. Secondly, one checks that the co-Kleisli category \mathcal{D} satisfies the universal property of Theorem 2. In particular, given any bi-Cartesian closed functor $F: \mathcal{C} \rightarrow \mathcal{E}$ and any morphism $g: 1 \rightarrow FA$, the extension $\hat{F}: \mathcal{D} \rightarrow \mathcal{E}$ is given by

$$\hat{F}(A) = F(A) \quad \text{and} \quad \hat{F}(f) = F(f) \circ \langle \text{Id}, g \rangle.$$

\square

In the rest of the paper we will only consider indeterminates $\xi: 1 \rightarrow A$. We write ‘ ξ is an indeterminate of type A ’ for such an indeterminate.

Next we show how to use polynomial categories to model reductions. The idea is that a reduction with non-atomic leaves $\Gamma_i \vdash \phi_i$ for $1 \leq i \leq n$ is an element of the category $\mathcal{C}(\xi_1, \dots, \xi_n)$, where $\mathcal{C}(\xi_1, \dots, \xi_n)$ is the category \mathcal{C} with indeterminates of type $\llbracket \Gamma_i \rrbracket \Rightarrow \llbracket \phi_i \rrbracket$ adjoined, where \Rightarrow denotes the internal hom.⁸ If \mathcal{C} is the free bi-Cartesian closed category over an infinite set of basic objects representing the propositional atoms, then there exists a morphism $1 \rightarrow \llbracket \phi \rrbracket$ in \mathcal{C} if and only if the formula ϕ is provable in LJ. If \mathcal{C} is not the free category, a morphism $1 \rightarrow \llbracket \phi \rrbracket$ exists in \mathcal{C} if the formula ϕ is provable in LJ with possible non-logical axioms added.

Each reduction operator is interpreted as a functor between the appropriate polynomial categories, and we show that a reduction is completeable to a proof when there exist morphisms f_i in $\mathcal{C}(\xi_1, \dots, \xi_{i-1})$ such that there is a functor $S_{\xi_1}(f_1) \circ \dots \circ S_{\xi_n}(f_n): \mathcal{C}(\xi_1, \dots, \xi_n) \rightarrow \mathcal{C}$.

⁸Note that we use indeterminates to witness reductions for arbitrary leaves rather than just atomic leaves.

Before we can state the semantics of LJ reductions we fix some notation about categorical morphisms. Suppose $f: A \times B \times C \rightarrow D$ is a morphism in \mathcal{C} . Then we denote by $\text{Cur}_B(f): A \times C \rightarrow B \Rightarrow D$ the morphism obtained by applying the definition of exponentials to f . We denote by App the morphism $A \times (A \Rightarrow B) \rightarrow B$. Furthermore, we denote the projections by $\pi: A \times B \rightarrow B$ and $\pi': A \times B \rightarrow A$ respectively. More generally, projections are denoted by $\pi_{A_i}: A_1 \times \dots \times A_i \times \dots \times A_m \rightarrow A_i$.

Before we give the definition of the translation from LJ-sequent reductions into morphisms in the polynomial category, we present an example. To state the example and the translation, for each indeterminate ξ of type $\llbracket \Gamma \rrbracket \Rightarrow \llbracket \phi \rrbracket$, ξ' denotes the morphism $\text{App} \circ \langle \xi \circ \langle \rangle, \text{Id} \rangle: \llbracket \Gamma \rrbracket \rightarrow \llbracket \phi \rrbracket$.

The morphism for the sequent reduction

$$\frac{\phi \text{ ?- } \phi \quad \phi, \psi \text{ ?- } \psi}{\phi, \phi \supset \psi \text{ ?- } \psi} \supset L$$

will be interpreted by a functor $H: \mathcal{C}(\xi) \rightarrow \mathcal{C}$, where ξ is the indeterminate of type $\llbracket (\phi \wedge (\phi \supset \psi)) \rrbracket$. In fact, H is the substitution functor $S_\xi(\text{Cur}_{\llbracket \phi \wedge (\phi \supset \psi) \rrbracket}(\text{App}))$. This functor arises in two stages. Firstly, we have the functor F with domain $\mathcal{C}(\xi)$ and co-domain $\mathcal{C}(\xi_1, \xi_2)$, where ξ_1 is an indeterminate of type $\llbracket \phi \supset \phi \rrbracket$ and ξ_2 and indeterminate of type $\llbracket \phi \wedge \psi \supset \psi \rrbracket$ respectively, such that $F = S_\xi(\xi'_2 \circ \langle \pi, \text{App} \circ \langle \pi', \xi'_1 \circ \pi \rangle \rangle)$. The functor F is the semantics of the inference rule $\supset L$, which is basically the application morphism and describes how to obtain a reduction for the sequent $\phi, \phi \supset \psi \text{ ?- } \psi$ corresponding to the indeterminate ξ from the two reductions for the sequents $\phi \text{ ?- } \phi$ and $\phi, \psi \text{ ?- } \psi$ corresponding to the indeterminates ξ_1 and ξ_2 . As the reductions for the latter two sequents are axioms, they are represented by the functors $G_1 = S_{\xi_1}(\text{Cur}_{\llbracket \phi \rrbracket}(\text{Id}))$ and $G_2 = S_{\xi_2}(\text{Cur}_{\llbracket \phi \wedge \psi \rrbracket}(\pi_{\llbracket \psi \rrbracket}))$. The functor H is obtained by essentially composing F with G_1 and G_2 .

After this example we give the definition of the translation.⁹

Definition 4 Let \mathcal{C} be a bi-Cartesian closed category. The interpretation of each unary LJ reduction operator

$$\frac{\Delta \text{ ?- } \psi}{\Gamma \text{ ?- } \phi}$$

in \mathcal{C} is a functor $\mathcal{C}(\xi) \rightarrow \mathcal{C}(\zeta)$, where ξ is an indeterminate of type $\llbracket \Gamma \rrbracket \Rightarrow \llbracket \phi \rrbracket$ and ζ is an indeterminate of type $\llbracket \Delta \rrbracket \Rightarrow \llbracket \psi \rrbracket$. The interpretation of a binary reduction operator

$$\frac{\Delta_1 \text{ ?- } \psi_1 \quad \Delta_2 \text{ ?- } \psi_2}{\Gamma \text{ ?- } \phi}$$

in \mathcal{C} is a functor $\mathcal{C}(\xi) \rightarrow \mathcal{C}(\xi_1, \xi_2)$, where ξ_1 , ξ_2 and ξ are indeterminates of types $\llbracket \Delta_1 \rrbracket \Rightarrow \llbracket \psi_1 \rrbracket$, $\llbracket \Delta_2 \rrbracket \Rightarrow \llbracket \psi_2 \rrbracket$ and $\llbracket \Gamma \rrbracket \Rightarrow \llbracket \phi \rrbracket$ respectively. These functors are defined as follows:

Axiom: If the reduction operator is

$$\overline{\Gamma, \phi \text{ ?- } \phi},$$

then $\llbracket Ax \rrbracket = S_\xi(\text{Cur}_{\llbracket \Gamma \rrbracket \times \llbracket \phi \rrbracket}(\pi_{\llbracket \phi \rrbracket}))$;

Cut: If the reduction operator is

$$\frac{\Gamma, \psi \text{ ?- } \phi \quad \Gamma \text{ ?- } \psi}{\Gamma, \text{ ?- } \phi}$$

and ξ , ξ_1 and ξ_2 are indeterminates of type $\llbracket \Gamma \rrbracket \Rightarrow \llbracket \phi \rrbracket$, $\llbracket \Gamma, \psi \rrbracket \Rightarrow \llbracket \phi \rrbracket$ and $\llbracket \Gamma \rrbracket \Rightarrow \llbracket \psi \rrbracket$ respectively, then we have $\llbracket Cut \rrbracket = S_\xi(\text{Cur}(\xi'_2 \circ (\text{Id}, \xi'_1)))$;

⁹Note that we include a clause for the Cut-rule. We need it for the completeness of the categorical semantics we are considering later in this chapter.

ExchangeL: If the reduction operator is

$$\frac{\Gamma, \phi_2, \phi_1 \text{ ?- } \phi}{\Gamma, \phi_1, \phi_2 \text{ ?- } \phi},$$

then $\llbracket \text{ExchangeL} \rrbracket = S_\xi(\text{Cur}_{\llbracket \Gamma, \phi_1, \phi_2 \rrbracket}(\zeta' \circ \langle \pi_{\llbracket \Gamma \rrbracket}, \pi_{\llbracket \phi_2 \rrbracket}, \pi_{\llbracket \phi_1 \rrbracket} \rangle))$;

$\perp L$: If the reduction operator is

$$\overline{\Gamma, \perp \text{ ?- } \phi},$$

then $\llbracket \perp L \rrbracket = \text{Cur}_{\llbracket \Gamma \rrbracket \times 0}(S_\xi(\iota \circ \pi))$, where ι is the initial morphism $0 \rightarrow \llbracket \phi \rrbracket$ and π is the projection from $\llbracket \Gamma \rrbracket \times 0$ to 0 ;

$\top R$: If the reduction operator is

$$\overline{\Gamma \text{ ?- } \top},$$

then $\llbracket \top R \rrbracket = \text{Cur}_{\llbracket \Gamma \rrbracket}(S_\xi(!))$, where $!$ is the unique morphism with the terminal object 1 as the co-domain;

$\wedge L$: $\llbracket \wedge L \rrbracket = S_\xi(\zeta)$;

$\wedge R$: If the reduction operator is

$$\frac{\Gamma \text{ ?- } \phi \quad \Gamma \text{ ?- } \psi}{\Gamma \text{ ?- } \phi \wedge \psi}$$

and ξ , ξ_1 and ξ_2 are indeterminates of type $\llbracket \Gamma \rrbracket \Rightarrow (\llbracket \phi \rrbracket \times \llbracket \psi \rrbracket)$, $\llbracket \Gamma \rrbracket \Rightarrow \llbracket \phi \rrbracket$ and $\llbracket \Gamma \rrbracket \Rightarrow \llbracket \psi \rrbracket$ respectively, then we have

$$\llbracket \wedge R \rrbracket = S_\xi(f \circ \langle \xi_1, \xi_2 \rangle)$$

where f is the canonical morphism with domain $(\llbracket \Gamma \rrbracket \Rightarrow \llbracket \phi \rrbracket) \times (\llbracket \Gamma \rrbracket \Rightarrow \llbracket \psi \rrbracket)$ and codomain $\llbracket \Gamma \rrbracket \Rightarrow (\llbracket \phi \rrbracket \times \llbracket \psi \rrbracket)$;

$\vee L$: If the reduction operator is

$$\frac{\Gamma, \phi \text{ ?- } \sigma \quad \Gamma, \psi \text{ ?- } \sigma}{\Gamma, \phi \vee \psi \text{ ?- } \sigma}$$

and ξ_1 and ξ_2 are indeterminates of type $(\llbracket \Gamma \rrbracket \times \llbracket \phi \rrbracket) \Rightarrow \llbracket \sigma \rrbracket$ and $(\llbracket \Gamma \rrbracket \times \llbracket \psi \rrbracket) \Rightarrow \llbracket \sigma \rrbracket$ respectively, then we have

$$\llbracket \vee L \rrbracket = S_\xi(f \circ (\text{Cur}_{\llbracket \phi \rrbracket + \llbracket \psi \rrbracket}(\text{Cur}_{\llbracket \Gamma \rrbracket}(\xi'_1) + \text{Cur}_{\llbracket \Gamma \rrbracket}(\xi'_2))))$$

where f is the canonical isomorphism between $(\llbracket \phi \rrbracket + \llbracket \psi \rrbracket) \Rightarrow \llbracket \Gamma \rrbracket \Rightarrow \llbracket \sigma \rrbracket$ and $(\llbracket \Gamma \rrbracket \times (\llbracket \phi \rrbracket + \llbracket \psi \rrbracket)) \Rightarrow \llbracket \sigma \rrbracket$;

$\vee R$: If the reduction operator is

$$\frac{\Gamma \text{ ?- } \phi}{\Gamma \text{ ?- } \phi \vee \psi}$$

and suppose ζ is an indeterminate of type $\llbracket \Gamma \rrbracket \Rightarrow \llbracket \phi \rrbracket$, then we have

$$\llbracket \vee R \rrbracket = S_\xi(\text{Cur}_{\llbracket \Gamma \rrbracket}(\text{in}_1 \circ \zeta')).$$

The other case is similar;

$\supset L$: If the reduction operator is

$$\frac{\Gamma \text{ ?- } \phi \quad \Gamma, \psi \text{ ?- } \sigma}{\Gamma, \phi \supset \psi \text{ ?- } \sigma}$$

and ξ_1 , ξ_2 and ξ are indeterminates of type $\llbracket \Gamma \rrbracket \Rightarrow \llbracket \phi \rrbracket$, $(\llbracket \Gamma \rrbracket \times \llbracket \psi \rrbracket) \Rightarrow \llbracket \sigma \rrbracket$ and $(\llbracket \Gamma \rrbracket \times (\llbracket \phi \rrbracket \Rightarrow \llbracket \psi \rrbracket)) \Rightarrow \llbracket \sigma \rrbracket$ respectively, then we have

$$\llbracket \supset L \rrbracket = S_\xi(\xi'_2 \circ \langle \pi, \text{App} \circ \langle \pi', \xi'_1 \circ \pi \rangle \rangle);$$

$\supset R$: If the reduction operator is

$$\frac{\Gamma, \phi \text{ ?- } \psi}{\Gamma \text{ ?- } \phi \supset \psi}$$

and ζ is an indeterminate of type $([\Gamma] \times [\phi]) \Rightarrow [\psi]$, then we have

$$[\supset R] = S_\xi(\text{Cur}_{[\Gamma]}(\text{Cur}_{[\phi]}(\zeta'))).$$

The interpretation of an LJ reduction $R_1; \dots; R_k$ for $\Gamma \text{ ?- } \phi$, where $;$ denotes the composition of operators and where the non-axiom leaves are $\Gamma_i \text{ ?- } \psi_i$ ($0 \leq i$), is given by a functor $H: \mathcal{C}(\xi) \rightarrow \mathcal{C}(\xi_1, \dots, \xi_n)$, where ξ is an indeterminate of type $[\Gamma] \Rightarrow [\phi]$ and ξ_i are indeterminates of type $[\Gamma_i] \Rightarrow [\psi_i]$ defined inductively as follows:

- If $k = 1$, then H is the interpretation of the reduction operator R_1 ;
- If $k > 1$ and $R_1; \dots; R_{k-1}$ is inductively interpreted as a functor

$$H: \mathcal{C}(\xi) \rightarrow \mathcal{C}(\xi_1, \dots, \xi_l, \eta)$$

and the reduction operator R_k is interpreted as the substitution functor $S_\eta(f)$ for some indeterminate η and morphism f in $\mathcal{C}(\xi_1, \dots, \xi_l)$, then the reduction $R_1; \dots; R_k$ is interpreted as the functor $G \circ H$, where G is the functor obtained by the universal property of polynomial categories applied to the maps

$$\xi_i \mapsto \xi_i \quad \text{and} \quad \eta \mapsto f.$$

An example will help to explain how this definition works. Consider the following reduction, which has one non-axiom leaf $\sigma, \tau \text{ ?- } \phi$:

$$\frac{X \quad \frac{\frac{\sigma, \psi \text{ ?- } \sigma \quad Ax}{\sigma, \sigma \supset \tau, \psi \text{ ?- } \tau} \quad \frac{\sigma, \psi, \tau \text{ ?- } \tau \quad Ax}{\sigma, \sigma \supset \tau, \psi \text{ ?- } \tau}}{\sigma, \phi \supset \psi, \sigma \supset \tau \text{ ?- } \tau} \supset L$$

where X is the reduction

$$\frac{\frac{\sigma \text{ ?- } \sigma \quad Ax}{\sigma, \sigma \supset \tau \text{ ?- } \phi} \quad \sigma, \tau \text{ ?- } \phi}{\sigma, \sigma \supset \tau \text{ ?- } \phi} \supset L.$$

If π_σ denotes the projection with co-domain $[\sigma]$, then the semantics of the reduction X is the morphism

$$S_{\xi_1}(\text{Cur}_{[\sigma] \times [\sigma \supset \tau]}(\chi' \circ \langle \pi_\sigma, \text{App}_{\sigma, \tau} \rangle))$$

where χ is an indeterminate of type $[\sigma \wedge \tau] \Rightarrow [\phi]$, and the semantics for the reduction of the sequent $\sigma, \sigma \supset \tau \text{ ?- } \tau$ is the morphism

$$S_{\xi_2}(\text{Cur}_{[\sigma] \times [\sigma \supset \tau]}(\text{App} \circ \langle \pi_\sigma, \pi_{\sigma \supset \tau} \rangle)).$$

The semantics for the whole derivation is then

$$S_\xi(\text{Cur}_{[\Gamma]}(\text{App}_{\sigma, \tau} \circ \langle \pi_\sigma, \pi_{\sigma \supset \tau} \rangle \circ \langle \pi_\sigma, \pi_{\sigma \supset \tau}, \text{App}_{\phi, \psi} \circ \langle \chi' \circ \langle \pi_\sigma, \text{App}_{\sigma, \tau} \rangle \circ \langle \pi_\sigma, \pi_{\sigma \supset \tau} \rangle \rangle, \pi_{\phi \supset \psi} \rangle))$$

which is via projection-equalities equal to

$$S_\xi(\text{Cur}_{[\Gamma]}(\text{App}_{\sigma, \tau} \circ \langle \pi_\sigma, \pi_{\sigma \supset \tau} \rangle)),$$

where Γ is the context $\sigma, \phi \supset \psi, \sigma \supset \tau$. This is unsatisfactory: the left-hand side of the reduction is ignored in the semantics; in other words any reduction with the same right-hand side but a different left-hand side has the same semantics.

The problem is that our semantics implicitly uses a translation from sequent calculus into natural deduction, as there is a direct correspondence between introduction and elimination rules of natural deduction and the categorical constructions. As the translation of sequent calculus into natural deduction identifies sequent calculus derivations up to certain permutations and some Cut-eliminations (see [41] for details), some sequent calculus derivations have the same semantics. In this particular case, the translation of $\supset L$ into natural deduction makes essential use of a cut. Because it is a cut with a weakened formula, after Cut-elimination the two derivations have identical natural deduction translation, and hence identical semantics.

2.1 Intuitionistic Reduction Models

We solve the problem of information loss described above by introducing a Kripke-world structure in which worlds are intended to record the history of application of reduction operators. Hence each application of a reduction operator gives rise to an extension of worlds. In the key case (*cf.* the example above) of $\supset L$, worlds may therefore be seen as recording increasing propositional ‘knowledge’ in hypotheses (or, in sequents, antecedents).

In [33], we used the $\lambda\mu\nu\epsilon$ -calculus, which is the $\lambda\mu\nu$ -calculus with explicit substitutions added, as a calculus of realizers for LK' -derivations. We added the explicit substitutions to overcome the same problem of information loss in a syntactic way. It is possible to treat the explicit substitutions semantically via a Kripke-world structure where the worlds do not contain all reduction operators but reductions corresponding to sufficient premisses of the $\supset L$ and $\forall L$ -rules which gave rise to explicit substitutions. The setting described below is more uniform, as it regards application of all reduction operators as an increase of knowledge. This is certainly appropriate for models of proof-search.

The categorical model we use to model this Kripke-world structure is a variant of the setting of a categorical semantics for intuitionistic logic based on functor categories. For deductions, one considers an indexed category with comprehension $F: \mathcal{W} \rightarrow CCC$, where \mathcal{W} is a partial order of worlds regarded as a category, and the functor F assigns to each world W a category $F(W)$ which models all derivations which have additional assumptions given by W . In our setting, worlds represent histories of which reductions have been applied.¹⁰ Hence we modify this semantics to require that the co-domain of the functor F is not the category of bi-Cartesian closed categories but rather a category which represents indeterminates. For each set of indeterminates, we require a bi-Cartesian closed category which models all reductions which use that set of indeterminates. An appropriate categorical structure for this modelling of indeterminates is given by an indexed category with comprehension. The base category models the indeterminates and the fibre over an object models the polynomial category over the indeterminates corresponding to this object. The universal properties of comprehension correspond to the universal property of the polynomial categories. The notion of an indexed category with comprehension is as follows:

Definition 5 *A strict indexed category with comprehension is a functor*

$$\mathcal{E}: \mathcal{B}^{op} \rightarrow \mathbf{Cat}$$

such that the following conditions are satisfied:

- (i) *\mathcal{B} has a terminal object called \top ;*
- (ii) *Each fibre $\mathcal{E}(\Gamma)$ has a terminal object 1 which is stable under re-indexing;*
- (iii) *If we denote by $Gr(\mathcal{E})$ the category whose objects are pairs (Γ, A) , where Γ is an object of \mathcal{B} and A an object of $\mathcal{E}(\Gamma)$, and morphisms from (Γ, A) to (Δ, B) are pairs of morphisms (f, g)*

¹⁰We can also think of worlds as representing the propositions which have been added to the hypotheses by the reduction, the key point being that the $\supset L$ -operator replaces a hypothesis $\phi \supset \psi$ with ψ , together with a proof obligation (for ϕ) which may be further reduced. This view is discussed briefly in [29].

where f is a morphism from Γ to Δ and g is a morphism from A to $\mathcal{E}(f)(B)$, then the functor $I: \mathcal{B} \rightarrow \text{Gr}(\mathcal{E})$ sending the object Γ to $(\Gamma, 1)$ and the morphism f to $(f, 1)$ has a right adjoint G .

We denote the object $G(\Gamma, A)$ by $\Gamma \cdot A$ and by $\langle f, g \rangle$ the part of the bijection between hom sets given by the adjunction $I \dashv G$ sending a morphism $f: \Gamma \rightarrow \Delta$ in \mathcal{B} and a morphism $g: 1 \rightarrow \mathcal{E}(f)A$ in $\mathcal{E}(\Gamma)$ to a morphism from Γ to $\Delta \cdot A$.

Now we explain how to set-up indeterminates in an indexed categorical setting. An indeterminate of type A is modelled by an object $\top \cdot A$, and a morphism in $\mathcal{C}(\xi_1, \dots, \xi_n)$ is modelled by a morphism in $\mathcal{E}(\top \cdot A_1 \cdot \dots \cdot A_n)$. The universal property of polynomial categories is captured as follows: if f is a morphism in $\mathcal{C}(\xi_1, \dots, \xi_n)$ corresponding to a morphism f' in $\mathcal{E}(\top \cdot A_1 \cdot \dots \cdot A_n)$ and ξ is an indeterminate of type A , the substitution functor $S_\xi(f)$ is modelled by the functor $\langle \text{Id}, f' \rangle$.

With all this technology set up, we can give a definition of a *reduction structure*, i.e., a semantic structure within which intuitionistic (LJ) reductions may be interpreted. A few points are noteworthy:

- As we have seen, the interpretation of LJ-reductions in polynomials over a bi-Cartesian closed category is inadequate. Consequently the interpretation of (Cut-free) LJ-reductions exploits a Kripke-world structure which records the history of the reduction;
- There is no equality in the semantics: We interpret only Cut-free reductions and do not consider any equality induced by Cut-elimination.

Definition 6 (reduction structure) Let \mathcal{W} be a small category (of ‘worlds’) with finite products. A reduction structure (\mathcal{E}, F) is given by

- (i) a strict indexed category $\mathcal{E}: \mathcal{B}^{op} \rightarrow \mathbf{Cat}$ with comprehension such that \mathcal{B} has finite products¹¹ and each fibre $\mathcal{E}(\Gamma)$ is a bi-Cartesian closed category and each functor $\mathcal{E}(f)$ preserves the bi-Cartesian closed structure on the nose;
- (ii) a functor $F: \mathcal{W} \rightarrow \mathcal{B}$ which preserves finite products.

Next we present a set-theoretic example of a reduction structure.

Example 7 (set-theoretic reduction structure) Let \mathcal{W} be the category of sets and functions. Let \mathcal{E} the indexed category arising from the flat fibration over \mathbf{Set} (i.e., \mathcal{B} is \mathbf{Set} again, and $\mathcal{E}(S)$ is the co-Kleisli category of \mathbf{Set} with respect to the functor $- \times \text{Id}_S$). We define the functor F as the identity functor from \mathbf{Set} to \mathbf{Set} .

Note that in this example, indeterminates and the state of knowledge given by worlds coincide, as the functor F is the identity. This is not necessarily true in general.

We now describe the interpretation of reduction operators and reductions in a reduction category. This interpretation depends on the worlds of the reduction category. The details are given in the following definition:

Definition 8 (interpretation) Let (\mathcal{E}, F) be a reduction structure. A function $\llbracket - \rrbracket$, which is parametrized by a list of indeterminates Θ and a world W , mapping reductions and their syntactic constituents to elements of a reduction structure is called an interpretation if it satisfies the following mutually recursive conditions:

- (i) $\llbracket \Theta \rrbracket^W$ is an object of \mathcal{B} and $\llbracket \Theta \rrbracket^W = A$ if Θ is the empty list of indeterminates and $F(W) = A$;
- (ii) For any formula ϕ , $\llbracket \phi \rrbracket_\Theta^W$ is an object of the category $\mathcal{E}(\llbracket \Theta \rrbracket^W)$;

¹¹If the functor $\mathcal{E}(f)$ is constant on objects then comprehension gives rise to finite products in \mathcal{B} . This is the case for all the reduction structures we consider in this paper.

- (iii) For any context $\Gamma = \phi_1, \dots, \phi_n$, $\llbracket \Gamma \rrbracket_{\Theta}^W$ is equal to $(A_1 \times \dots \times A_n)$, where $\llbracket \phi_i \rrbracket_{\Theta}^W = A_i$;
- (iv) For a reduction $\Phi; \Gamma \vdash \phi$ with indeterminates in Θ , $\llbracket \Phi \rrbracket_{\Theta}^W$ is a pair (W', g) , where W' is a world and g a morphism from $\llbracket \Gamma \rrbracket_{\Theta}^W$ to $\llbracket \phi \rrbracket_{\Theta}^W$ such that $g = \langle \text{Id}, F(a) \rangle^* f$ for some morphisms $f: \llbracket \Gamma \rrbracket_{\Theta}^{W'} \rightarrow \llbracket \phi \rrbracket_{\Theta}^{W'}$ and $a: W \rightarrow W'$;¹²
- (v) For all reduction operators R , there exists a world W_R and a morphism $a_R: 1 \rightarrow W_R$;
- (vi) For a reduction $\Phi; R$ with unary reduction operator R and reduction Φ for the putative premiss of R ,

$$\llbracket \Phi; R \rrbracket_{\Theta}^W = (W', \langle \text{Id}, F(a) \rangle^* (\text{Cur}_{\llbracket \Gamma \rrbracket_{\Theta}^{W \times W_R}}^{-1} (\text{App} \circ \langle \text{Cur}_{\llbracket \Gamma_1 \rrbracket_{\Theta}^{W \times W_R}}(f_1), \text{Snd}) \rangle))$$
 where $W' = W_1 \times W \times W_R$ and furthermore $\llbracket \Phi \rrbracket_{\Theta}^{W \times W_R} = (W'_1, f_1)$ and $W'_1 = W \times W_R \times W_1$ and $a: W \rightarrow W'$;
- (vii) For a reduction $(\Phi_1, \Phi_2); R$ with binary reduction operator R and reductions Φ_1 and Φ_2 for the putative premisses of R ,

$$\begin{aligned} \llbracket (\Phi_1, \Phi_2); R \rrbracket_{\Theta}^W &= (W', \langle \text{Id}, F(a_{W'}) \rangle^* (\text{Cur}_{\llbracket \Gamma \rrbracket_{\Theta}^{W \times W_R}}^{-1} \\ &\quad (\text{App} \circ \langle \langle \text{Cur}_{\llbracket \Gamma_1 \rrbracket_{\Theta}^{W \times W_R}}(f_1), \text{Cur}_{\llbracket \Gamma_2 \rrbracket_{\Theta}^{W \times W_R}}(f_2) \rangle, \text{Snd}) \rangle)) \end{aligned}$$
 where $W' = W_1 \times W_2 \times W \times W_R$ and $\llbracket \Phi_i \rrbracket_{\Theta}^{W \times W_R} = (W'_i, f_i)$ and $W'_i = W \times W_R \times W_i$;
- (viii) If $\Theta = \Theta', \xi$, where ξ is an indeterminate for $\phi_1, \dots, \phi_n \vdash \phi$, then $\llbracket \Theta \rrbracket^W$ is equal to $\llbracket \Theta' \rrbracket^W \cdot \llbracket (\phi_1 \wedge \dots \wedge \phi_n) \supset \phi \rrbracket_{\Theta'}^W$.

This definition only specifies which elements of a reduction structure are used to interpret a given syntactic constituent of a reduction: each reduction operator gives rise to a change of worlds (Clause (v)), and the functor F describes how extensions of worlds give rise to change of indeterminates corresponding to reduction operators. Clauses (vi) and (vii) say that the semantics of a reduction Φ is given by a pair (a, f) , where a is an extension of worlds induced by the reduction operators of Φ , and f is the morphism obtained by applying the changes of indeterminates induced by the reduction operators to the indeterminates representing the premisses of the reduction.

However, this definition does not specify how to interpret the logical connectives and operators in a reduction. As we use bi-Cartesian closed categories for interpreting reductions, this can be done in a canonical way for the logical connectives and operators of intuitionistic logic. In this way, we obtain a canonical interpretation which is a specific function from syntactic constituents of a reduction to elements of a reduction structure.

Definition 9 (canonical interpretation) Let (\mathcal{E}, F) be a reduction structure. The following function $\llbracket - \rrbracket$ is an interpretation, called the canonical interpretation, where Θ is a list of indeterminates:

- (i) $\llbracket \perp \rrbracket_{\Theta}^W \stackrel{\text{def}}{=} 0$;
- (ii) $\llbracket \top \rrbracket_{\Theta}^W \stackrel{\text{def}}{=} 1$;
- (iii) $\llbracket \phi \supset \psi \rrbracket_{\Theta}^W \stackrel{\text{def}}{=} \llbracket \phi \rrbracket_{\Theta}^W \Rightarrow \llbracket \psi \rrbracket_{\Theta}^W$;
- (iv) $\llbracket \phi \wedge \psi \rrbracket_{\Theta}^W \stackrel{\text{def}}{=} \llbracket \phi \rrbracket_{\Theta}^W \times \llbracket \psi \rrbracket_{\Theta}^W$;
- (v) $\llbracket \phi \vee \psi \rrbracket_{\Theta}^W \stackrel{\text{def}}{=} \llbracket \phi \rrbracket_{\Theta}^W + \llbracket \psi \rrbracket_{\Theta}^W$;
- (vi) For all reduction operators R , $F(a_R) = \langle \text{Id}_1, f \rangle$, where $S_{\xi}(f)$ is the interpretation of R according to Definition 4, where the category \mathcal{C} is the category $\mathcal{E}(1)$.

¹²Here $-^*$ denotes the usual inverse image functor.

Note that this definition ensures that for each world W_R the object $F(W_R)$ is the object $(\llbracket \Gamma_1 \rrbracket \Rightarrow \llbracket \phi_1 \rrbracket) \Rightarrow \llbracket \Gamma \rrbracket \Rightarrow \llbracket \phi \rrbracket$ for a unary reduction operator R with sufficient premiss $\Gamma_1 \text{ ?- } \phi_1$ and putative conclusion $\Gamma \text{ ?- } \phi$ and $F(W_R)$ is the object $((\llbracket \Gamma_1 \rrbracket \Rightarrow \llbracket \phi_1 \rrbracket) \times (\llbracket \Gamma_2 \rrbracket \Rightarrow \llbracket \phi_2 \rrbracket)) \Rightarrow (\llbracket \Gamma \rrbracket \Rightarrow \llbracket \phi \rrbracket)$ for a reduction operator R with sufficient premisses $\Gamma_1 \text{ ?- } \phi_1$ and $\Gamma_2 \text{ ?- } \phi_2$ and putative conclusion $\Gamma \text{ ?- } \phi$. For each reduction operator R , we denote by ξ_R the indeterminate of the above type, and with A_R the corresponding object.

As mentioned before, the interpretation does not enforce any equality between reductions: The reason is that the semantics of a reduction is a pair (f, g) , where f is a morphism between worlds, and it is possible that each reduction gives rise to a different morphism f . Two different reductions might give rise to the same morphism g however.

Now let us reconsider our earlier example. We need to be precise and indicate carefully the changes of the worlds involved in the reduction. We construct the semantics of the whole reduction, $\llbracket \Phi \rrbracket_\chi^1$, where χ is an indeterminate of type $\llbracket \sigma \wedge \tau \rrbracket \Rightarrow \llbracket \phi \rrbracket$ and A the corresponding object in the base category, step by step. We start with the reduction X . Following Clause (vii), we have to calculate $\llbracket X \rrbracket_\chi^{W_{\supset L}}$. We obtain $\llbracket X \rrbracket_\chi^{W_{\supset L}} = (a_X, f_X)$, where a_X is the extension of worlds from the world $W_{\supset L}$ to $W_{\supset L} \times W_{\supset L} \times W_{Ax}$ and f_X is the morphism

$$h \circ \langle \pi_\sigma, \text{App}_{\sigma, \tau} \rangle$$

in the fibre $E(A_{W_{\supset L}} \times A)$, where h is the morphism $\text{App} \circ \langle \text{Id}, \text{Snd} \circ ! \rangle$.

Next, we have to calculate $\llbracket \Phi \rrbracket_\chi^{W_{\supset L}}$, where Φ is the reduction of the sequent $\sigma, \sigma \supset \tau \text{ ?- } \tau$. Again, $\llbracket \Phi \rrbracket_\chi^{W_{\supset L}}$ is a pair (a_Φ, f_Φ) , where a_Φ is the extension of worlds from the world $W_{\supset L}$ to $W_{\supset L} \times W_{\supset L} \times W_{Ax} \times W_{Ax}$, and the morphism f_Φ is the morphism

$$\text{App} \circ \langle \pi_\sigma, \pi_{\sigma \supset \tau} \rangle$$

in the fibre $E(A_{W_{\supset L}} \times A)$. The semantics for the whole reduction Ψ is a pair (a_Ψ, f_Ψ) , where a_Ψ is the world extension from the empty world (the terminal object in the category \mathcal{W}) to the world $W_{\supset L} \times W_{\supset L} \times W_{Ax} \times W_{\supset L} \times W_{Ax} \times W_{Ax}$, and the morphism f_Ψ is the morphism

$$\text{App}_{\sigma, \tau} \circ \langle \pi_\sigma, \pi_{\sigma \supset \tau} \rangle \circ \langle \pi_\sigma, \pi_{\sigma \supset \tau}, \text{App}_{\phi, \psi} \circ \langle h \circ \langle \pi_\sigma, \text{App}_{\sigma, \tau} \circ \langle \pi_\sigma, \pi_{\sigma \supset \tau} \rangle \rangle, \pi_{\phi \supset \psi} \rangle \rangle$$

in the fibre $E(A)$, which is via projection-equalities equal to

$$\text{App}_{\sigma, \tau} \circ \langle \pi_\sigma, \pi_{\sigma \supset \tau} \rangle,$$

where Γ is the context $\sigma, \phi \supset \psi, \sigma \supset \tau$.

The semantics of the reduction Ψ does not ignore the reduction X : the world extension a_Φ explicitly mentions the reduction operators in X , thereby recording the increase of knowledge obtained by the reduction X .

Our objective has been to establish a semantics of reductive logic of comparable value to that which is available for deductive logic. To this end, we now establish soundness and completeness theorems relating reductions and their semantics. We begin with the appropriate semantic judgement,

$$W \models_\Theta (\Phi : \phi)[\Gamma],$$

between worlds, W , indeterminates in Θ , sequents $\Gamma \text{ ?- } \phi$ and reductions, Φ . This judgement is formulated as a constraint on reduction structures which is required in order to interpret reductions correctly in reduction structures.

Definition 10 (reduction model) A reduction model,

$$\mathcal{R} = \langle (\mathcal{E}, F), \llbracket - \rrbracket, \models \rangle,$$

is given by the following:

- A reduction structure (\mathcal{E}, F) ;
- An interpretation $\llbracket - \rrbracket$ of reduction operators and reductions;
- A forcing relation $W \models_{\Theta} (\Phi : \phi)[\Gamma]$, where W is a world, Θ and Γ are contexts, ϕ a formula and Φ a reduction with endsequent $\Gamma \text{ ?- } \phi$ with indeterminates contained in Θ , such that

$$\llbracket \Gamma \rrbracket_{\Theta}^W \xrightarrow{\llbracket \Phi \rrbracket_{\Theta}^W} \llbracket \phi \rrbracket_{\Theta}^W$$

is a morphism in the reduction structure, and which satisfies the following conditions:

1. If $W \models_{\Theta} (\Phi : \phi)[\Gamma]$ and $a: W \rightarrow W'$ is a morphism in \mathcal{W} for some world W' , then also $W' \models_{\Theta} (\Phi : \phi)[\Gamma]$;
2. $W \models_{\Theta} (Ax : \phi)[\Gamma, \phi]$;
3. $W \models_{\Theta, \xi} (\xi : \phi)[\Gamma]$ if ξ is an indeterminate of type $\Gamma \text{ ?- } \phi$;
4. If R is a reduction operator with premisses $\Gamma_1 \text{ ?- } \phi_1$ and $\Gamma_2 \text{ ?- } \phi_2$ and conclusion $\Gamma \text{ ?- } \phi$, then $W \models_{\Theta} ((\Phi_1, \Phi_2); R)[\Gamma, \phi]$ if $W \times W_R \models_{\Theta} (\Phi_i)[\Gamma_i, \phi_i]$;
5. If R is a reduction operator with premiss $\Gamma_1 \text{ ?- } \phi_1$ and conclusion $\Gamma \text{ ?- } \phi$, then $W \models_{\Theta} (\Phi_1; R)[\Gamma, \phi]$ if $W \times W_R \models_{\Theta} (\Phi_1)[\Gamma_1, \phi_1]$.

Substitutivity for indeterminates is a property of the forcing relation:

Lemma 11 *If $W \models_{\Theta, \xi} (\Phi : \phi)[\Gamma]$, $W \models_{\Theta} (\Psi : \psi)[\Delta]$ and ξ is an indeterminate of type $\Delta \text{ ?- } \psi$, then also $W \models_{\Theta} (\Phi[\Psi/\xi] : \phi)[\Gamma]$.*

Proof By induction over the structure of Φ . □

Now we can establish soundness: the existence of a reduction Φ of $\Gamma \text{ ?- } \phi$ implies that ϕ is forced at every world W in a reduction model and, consequently, that reduction Φ is interpreted as a realizer of the interpretation of ϕ from the interpretation of Γ .

Theorem 12 (soundness) *Consider any reduction structure (\mathcal{E}, F) . Suppose Φ is a reduction of $\Gamma \text{ ?- } \phi$ with indeterminates ξ_1, \dots, ξ_n of type $\Gamma_i \text{ ?- } \phi_i$. Then, for any world W , $W \models_{\Theta} (\Phi : \phi)[\Gamma]$, where $\Theta = \{\xi_1, \dots, \xi_n\}$.*

Proof We use induction over the structure of ϕ . The case of an indeterminate and an axiom are trivial. Now consider the case of a reduction $(\Phi_1, \Phi_2); R$. By induction hypothesis, $W \times W_R \models_{\Theta} (\Phi_i)[\Gamma_i, \phi_i]$. Hence, by Clause (iv) of Definition 10, $W \models_{\Theta} ((\Phi_1, \Phi_2); R)[\Gamma, \phi]$. The case of a unary reduction rule is similar. □

Turning to completeness, we must first establish a notion of *validity*. We say that the judgement $\Phi : \phi$ is *valid with respect to Γ and Θ* , and write

$$\Gamma \models_{\Theta} \Phi : \phi,$$

if and only if, for all worlds, W , in all reduction models, \mathcal{R} ,

$$W \models_{\Theta}^{\mathcal{R}} (\Phi : \phi)[\Gamma].$$

With respect this, quite straightforward, notion of validity, we are able to establish completeness. The first step is a model existence lemma based on the construction of a term model.

Lemma 13 (model existence) *There exists a reduction model*

$$\mathcal{T} = \langle F, \llbracket - \rrbracket, \models_{\Theta} \rangle,$$

such that if $1 \models_{\Theta}^{\mathcal{T}} (\Phi : \phi)[\Gamma]$, then $\Gamma \models_{\Theta} \Phi : \phi$.

Proof We construct a term model from reductions in the calculus LJ. We begin by defining a reduction structure (\mathcal{E}, F) .

The category of worlds is the free cartesian category where

- the ground objects are reduction operators R with sufficient premisses $\Gamma_1 \text{ ?- } \phi_1$ and $\Gamma_2 \text{ ?- } \phi_2$, for binary operators, and $\Gamma' \text{ ?- } \phi'$, for unary operators, and putative conclusion $\Gamma \text{ ?- } \phi$, and
- for each reduction operator, there is a ground morphism $a_R: 1 \rightarrow R$.

The objects of the category \mathcal{B} are finite sequences of indeterminates ξ_1, \dots, ξ_n of type ϕ_1, \dots, ϕ_n , and a morphism from (ξ_1, \dots, ξ_n) to (ξ'_1, \dots, ξ'_m) is a list (f_1, \dots, f_m) of reductions such that f_i is a reduction of $- \text{ ?- } \phi'_i$, possibly using the indeterminates ξ_1, \dots, ξ_n , where ϕ'_i is the type of the indeterminate ξ'_i . Composition is given by substitution of reductions for indeterminates. For each sequence of indeterminates (ξ_1, \dots, ξ_n) , we define the category $\mathcal{E}((\xi_1, \dots, \xi_n))$ to be the category where the objects are formulæ and morphisms from ϕ to ψ are reductions with premiss ϕ and conclusion ψ with indeterminates amongst the ones in (ξ_1, \dots, ξ_n) up to $\beta\eta$ -equivalence. Composition in this category is given by Cut. There is no equivalence on propositions, as we do not consider any type dependency.

As \mathcal{W} is the free cartesian category over the reduction operators R and morphisms a_R , it suffices to define the action of F on reduction operators and morphisms a_R .

For a binary reduction operator, the functor F is given by $F(R) = \xi$, where ξ is an indeterminate of type $((\Gamma_1 \supset \phi_1) \wedge (\Gamma_2 \supset \phi_2)) \supset (\Gamma \supset \phi)$,¹³ where the reduction operator with sufficient premisses $\Gamma_1 \text{ ?- } \phi_1$ and $\Gamma_2 \text{ ?- } \phi_2$ and putative conclusion $\Gamma \text{ ?- } \phi$ and $F(a_R) = h$, where $S_\xi(h)$ is the interpretation of R in the polynomial categories.

For a unary reduction operator, we define $F(R) = \xi$, where ξ is an indeterminate of type $(\Gamma' \supset \phi') \supset (\Gamma \supset \phi)$ for a reduction operator with sufficient premiss $\Gamma' \text{ ?- } \phi'$ and putative conclusion $\Gamma \text{ ?- } \phi$.

Now consider the morphism a_R for the reduction operator with sufficient premisses $\Gamma_1 \text{ ?- } \phi_1$ and $\Gamma_2 \text{ ?- } \phi_2$ and putative conclusion $\Gamma \text{ ?- } \phi$. Let Φ_R be sequent reduction

$$\frac{\frac{\frac{}{\Gamma_2 \supset \phi_2, \Gamma_1 \text{ ?- } \Gamma_1} Ax}{\Gamma_1 \supset \phi_1, \Gamma_2 \supset \phi_2, \Gamma_1 \text{ ?- } \phi_1} \supset L \quad \frac{\frac{\frac{}{\Gamma_2 \supset \phi_2, \Gamma_1, \phi_1 \text{ ?- } \phi_1} Ax}{\Gamma_1 \supset \phi_1, \Gamma_2 \supset \phi_2, \Gamma_2 \text{ ?- } \phi_2} \Psi}{\Gamma_1 \supset \phi_1, \Gamma_2 \supset \phi_2, \Gamma \text{ ?- } \phi} R}{\Gamma_1 \supset \phi_1, \Gamma_2 \supset \phi_2, \Gamma \text{ ?- } \phi} \wedge L; \supset R; R$$

where Ψ is the sequent reduction of $\Gamma_1 \supset \phi_1, \Gamma_2 \supset \phi_2, \Gamma_2 \text{ ?- } \phi_2$ similar to the reduction of $\Gamma_1 \supset \phi_1, \Gamma_2 \supset \phi_2, \Gamma_1 \text{ ?- } \phi_1$. Now we define $F(a_R) = \Phi_R$. Intuitively, the additional reduction steps in Φ_R are just book-keeping steps to ensure the typing of $F(a_R)$ matches the typing of the corresponding indeterminate.

Next, we show that, for this reduction structure with the obvious interpretation $\llbracket - \rrbracket$ of operators and reductions, the relation defined by $W \models_\Theta (\Phi: \phi)[\Gamma]$ iff Φ is a reduction of $\Gamma \text{ ?- } \phi$ with indeterminates in Θ such that $\llbracket \Phi \rrbracket_\Theta^W$ is a morphism from $\llbracket \Gamma \rrbracket_\Theta^W$ to $\llbracket \phi \rrbracket_\Theta^W$ is a forcing relation, and the triple $((\mathcal{E}, F), \llbracket - \rrbracket, \models)$ is a reduction model. \square

In the usual way, we now obtain the following:

Theorem 14 (completeness) *If $\Gamma \models_\Theta \Phi: \phi$, then $\Gamma \text{ ?- } \Phi: \phi$.*

¹³Here we abuse notation slightly and write, where $\Gamma = \psi_1, \dots, \psi_m$, just $\Gamma \supset \phi$ to denote the formula $(\psi_1 \wedge \dots \wedge \psi_m) \supset \phi$.

Proof Suppose $\Gamma \models_{\Theta} \Phi : \phi$, then, for all worlds W in all reduction models \mathcal{R} ,

$$W \models_{\Theta}^{\mathcal{R}} (\Phi : \phi)$$

This holds also for the term model constructed in Lemma 13. By construction of this model, we have $\Gamma \vdash_{\Theta} \Phi : \phi$. \square

Under stronger conditions, namely that there exists a canonical interpretation, we can show more, namely for each reduction structure and interpretation there exists a canonical forcing relation:

Lemma 15 *Suppose (\mathcal{E}, F) is a reduction structure with a canonical interpretation $\llbracket - \rrbracket$. Then the relation \mathcal{R} defined by $W \models_{\Theta} (\Phi : \phi) [\Gamma]$ iff Φ is a reduction of $\Gamma \vdash_{\Theta} \phi$ with indeterminates in Θ such that $\llbracket \Phi \rrbracket_{\Theta}^W$ is a morphism from $\llbracket \Gamma \rrbracket_{\Theta}^W$ to $\llbracket \phi \rrbracket_{\Theta}^W$ is a forcing relation, and the triple $((\mathcal{E}, F), \llbracket - \rrbracket, \models)$ is a reduction model. Moreover, the forcing relation \models of Lemma 13 is such a relation.*

Proof We have to check that the relation \mathcal{R} is a forcing relation. For this, one shows that Clause (vi) of Definition 9 implies that $\llbracket \Phi \rrbracket_{\Theta}^W$ is indeed a morphism from $\llbracket \Gamma \rrbracket_{\Theta}^W$ to $\llbracket \phi \rrbracket_{\Theta}^W$. \square

Now we consider the translation in the other direction. As the reduction category is not necessarily the free category over some ground objects, we cannot define such a translation inductively but only specify constraints which such a translation should satisfy. If the reduction structure happens to a free structure, the conditions turn out to define a translation uniquely. We define this translation first for polynomial categories and then generalize it to reduction structures.

Definition 16 *Let \mathcal{C} be any bi-Cartesian category. A translation $(-)^s$ assigning morphisms $f : \Gamma \rightarrow A$ in $\mathcal{C}(\xi_1, \dots, \xi_n)$ to reductions with non-atomic endsequents contained in $\xi_i : \Gamma_i \vdash_{\Theta} A_i$ is called sound if:*

$$\begin{aligned} (\pi)^s &= Ax, \text{ where } \pi \text{ is any projection} \\ (\xi_i)^s &= \xi_i \\ (g \circ f)^s &= ((f)^s, (g)^s); Cut \\ (\langle f, g \rangle)^s &= ((f)^s, (g)^s); \wedge R \\ (Cur M)^s &= (M)^s; \supset R \\ (App)^s &= \supset L \\ (in_1)^s &= \vee R \\ (in_2)^s &= \vee R \\ (f \oplus g)^s &= ((f)^s, (g)^s); \vee L \end{aligned}$$

Lemma 17 *Suppose \mathcal{C} is the free bi-Cartesian closed category over some set of objects \mathcal{G} . Then there is a sound translation assigning to each morphism $f : \Gamma \rightarrow A$ in $\mathcal{C}(\xi_1, \dots, \xi_n)$ reductions with non-atomic endsequents contained in $\xi_i : \Gamma_i \vdash_{\Theta} A_i$.*

Proof The translation is given in the canonical way by using the Curry–Howard correspondence to derive natural deductions for morphisms, and then translating them into reductions. \square

Now we generalize this translation to the translation of morphisms of reduction structures to reductions. Again, we list first conditions which such a translation should satisfy.

Definition 18 *A translation $(-)^s$ from morphisms $f : \Gamma \rightarrow A$ in $\mathcal{E}(\Theta)$ of a reduction structure (\mathcal{E}, F) to reductions $\Gamma \rightarrow A$ where $\Theta = \top \cdot A_1 \cdots A_n$ and ξ_i is an indeterminate with type $(A_i)^s$ is sound if*

$$\begin{aligned} (\pi)^s &= Ax, \text{ where } \pi \text{ is any projection} \\ (Fst^k * Snd)^s &= \xi_k \end{aligned}$$

$$\begin{aligned}
(g \circ f)^s &= ((f)^s, (g)^s); Cut \\
(\langle f, g \rangle)^s &= ((f)^s, (g)^s); \wedge R \\
(Cur M)^s &= (M)^s; \supset R \\
(App)^s &= \supset L \\
(in_1)^s &= \vee R \\
(in_2)^s &= \vee R \\
(f \oplus g)^s &= ((f)^s, (g)^s); \vee L
\end{aligned}$$

Again, when we have the initial reduction structure these conditions are sufficient to guarantee the existence of such a translation.

Lemma 19 *Suppose (\mathcal{E}, F) is a reduction structure such that $Gr(\mathcal{E})$ is the free comprehension category over some set of objects \mathcal{G} . Then there is a sound translation assigning to each morphism $f: \Gamma \rightarrow A$ in $\mathcal{E}(\Theta)$ a reduction with non-atomic endsequents contained in $\xi_i: \Gamma_i \vdash A_i$, where Θ is the context corresponding to the indeterminates ξ_1, \dots, ξ_n .*

Proof Direct transfer of the previous lemma. □

Now we can show that a reduction can be completed if and only if there exists a functor from the corresponding polynomial category into the ground category.

Theorem 20 *Suppose (\mathcal{E}, F) is the free reduction structure over a set of objects \mathcal{G} . A reduction Φ of $\Gamma \vdash \phi$ with leaves $\Gamma_i \vdash \phi_i$ which are not axioms can be completed to a proof iff there exists a morphism f such that there is a functor $\mathcal{E}(!, f): \mathcal{E}(\Theta) \rightarrow \mathcal{E}(1)$ where Θ is the context corresponding to the indeterminates ξ_1, \dots, ξ_n . Moreover, the completion of a Cut-free reduction is Cut-free.*

Proof If there exists a completion, the soundness theorem guarantees the existence of a morphism f . In the other direction, given such a morphism, the previous lemma provides the sequent derivations which complete the reduction to a proof. As Cut-elimination holds for LJ without indeterminates, there is also a cut-free sequent which provides the completion. □

2.2 Games for Intuitionistic Reductions

In this section we present a games model of intuitionistic reductions. Games models have been used successfully as models for various computational effects. We present here a version which will turn out to be a suitable basis for games models of both reductive intuitionistic logic and proof-search.

We consider games played between two players, Proponent, P , and Opponent, O . In such games for a formula ϕ the aim of Opponent is to falsify the given formula ϕ , and the aim of Proponent is to prove it. A game starts by Opponent challenging the given formula. Proponent wins a game when he can answer Opponent's initial challenge, otherwise he loses. The possible moves of both players in a game for ϕ are determined by the structure of ϕ . A proof of a formula corresponds to a *winning strategy* for Proponent. Such a winning strategy for a formula ϕ is a function which for every legal O-move in a game for ϕ produces a legal P-move such that if P uses this strategy to determine his moves he wins every game for ϕ . Such games for proofs have been described for a variety of logics, including classical and intuitionistic logic [21, 5]. Usually, in games for classical logic Proponent and Opponent are dual to each other, whereas this is not true for games for intuitionistic logic.

These game models for proofs have been adapted to give models of sequential computations in programming languages [13, 1, 2, 22]. Here, the intuition is that Opponent asks for the value of a computation, and Proponent performs the computation to produce values as answers. In such games there is usually a strict alternation between moves by Proponent and Opponent, corresponding to the absence of concurrent computation. As computations have a clear direction

(from inputs to outputs) there is usually no duality between Proponent and Opponent in these games.

The key conceptual difference between the games for proofs and the games for computations is that in logic not all propositions are provable, so that in these games not all propositions have strategies, whereas in the programming languages considered, however, all types are inhabited, so that these games have strategies for every type.

The details of how to present game models differ widely, both within games for proofs and within games for computations. The definition of the games considered in this paper uses elements of both approaches. We use one important technical notion from the games introduced by Hyland and Ong, namely the notion of an arena: for each formula ϕ the possible moves for a game for ϕ are listed in a forest¹⁴ called an arena, and the rules of the game use this forest extensively. Ong [23] introduces also the notion of a *scratchpad* to model the multiple conclusions in the $\lambda\mu$ -calculus. Scratchpads are additional games, which Proponent may start at will. For a detailed explanation of these scratchpads, see § 4.

This idea of games semantics in the context of proof theory was introduced by Lorenzen [21, 5]. For games semantics as a semantics of programming languages see [13, 2, 1, 22]. A comprehensive summary is provided in [16]. The use of game-theoretic methods in model theory, however, has a rather longer history, beginning with Ehrenfeucht-Fraïssé games, in which the *back and forth equivalence* of models is used to analyse completeness properties of (first-order) theories [11].

Hyland [16] provides a useful general comparison, in terms of categorical composition, of the correspondences between λ -calculus, proofs, algorithms and strategies:

Object	Map	Composition
Type	Proof	Application in context
Proposition	Proof	Composition via the Cut rule
Type	Algorithm	Composition with hiding
Game	Strategy	Scratchpad composition

This organization captures the main themes of this paper, all of which are expressed within the structures of categorical logic:

- The propositions-as-types (Curry-Howard-de Bruin) correspondence;
- The programs-as-proofs correspondence; and
- Games as a semantics for both proofs and computations.

Now we introduce a particular class of games which combines ideas from those for intuitionistic provability and those for programming languages to give a class which models intuitionistic proofs *directly*.¹⁵ Moreover, our games extend cleanly not only to the semantics of classical proofs provided by models of the $\lambda\mu\nu$ -calculus, described in § 3.2 but also to the structures required to interpret reductive logic and proof-search.

We start the definition of our games semantics by defining arenas. For each formula ϕ we define an arena, which is a forest used to characterize legal moves by both players in our games.

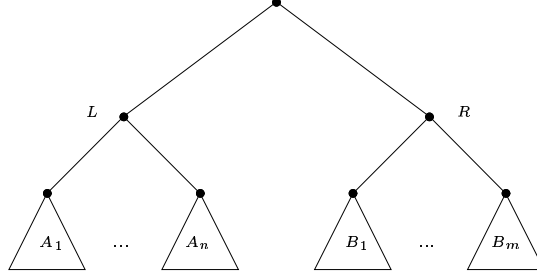
Definition 21 *An arena of type ϕ is a forest with nodes having possibly labels defined inductively by the following:*

- The arena of \top is the empty forest;
- The arena of \perp is the forest with one node labelled \perp ;

¹⁴A *forest* is a set of trees.

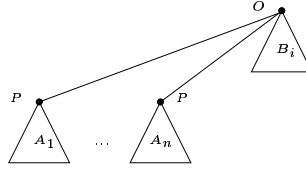
¹⁵Games models of intuitionistic proof can be recovered from games models of linear proofs [2] via the exponential ! and, for example, Girard's translation of intuitionistic logic into linear logic.

- The arena for a propositional atom p is a forest with one node labelled p ;
- The arena for $\phi \wedge \psi$ is the disjoint sum of the arenas for ϕ and ψ ;
- Suppose $\mathcal{A}_1, \dots, \mathcal{A}_n$ are the trees of the arena for ϕ and $\mathcal{B}_1, \dots, \mathcal{B}_m$ are the trees of the arena for ψ . Then the arena for $\phi \vee \psi$ is given by



Note that there are two special nodes called L and R . In the special case that the arena for ϕ or the arena for ψ is empty, the arena for $\phi \vee \psi$ is the empty arena too. The root node of the arena for $\phi \vee \psi$ is labelled \vee ;

- Suppose $\mathcal{A}_1, \dots, \mathcal{A}_n$ are the trees of the arena for ϕ and $\mathcal{B}_1, \dots, \mathcal{B}_m$ are the trees of the arena for ψ . Then the arena for $\phi \supset \psi$ is the disjoint union of the following trees



In the special case that the arena for ϕ is empty, the arena for $\phi \supset \psi$ is the arena for ψ . All nodes in the arena for $\phi \supset \psi$ which are root nodes in the arena of ψ are labelled \supset in addition to any other label they might have.

We call all root nodes in an arena O -nodes, and all children of O -nodes P -nodes, and all children of P -nodes O -nodes.

Arenas are used to define possible plays. The definition of moves and plays makes this precise.

We illustrate games for intuitionistic proofs using the formula

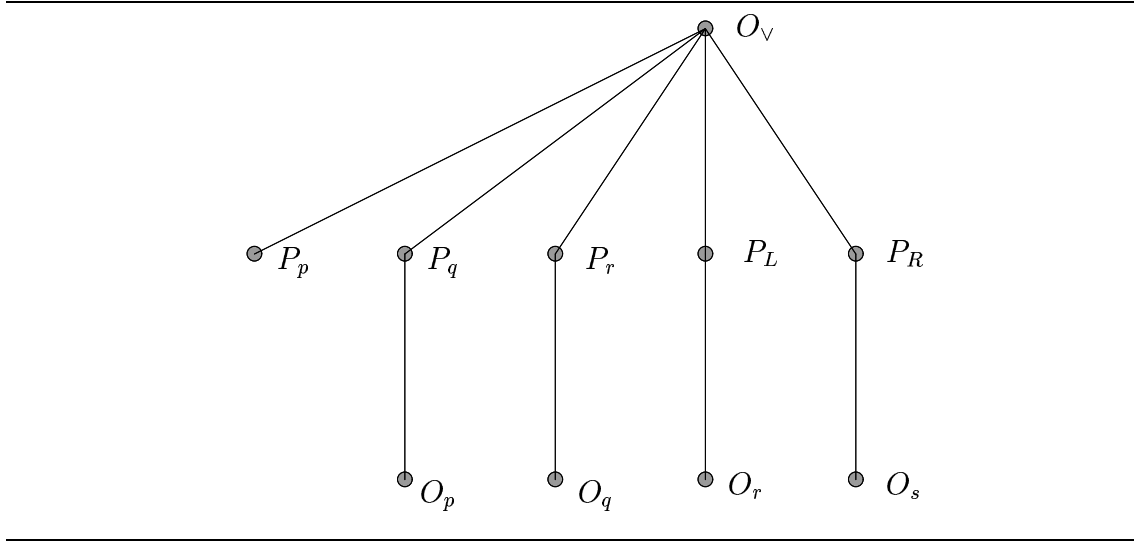
$$p \supset (p \supset q) \supset (q \supset r) \supset (r \vee s).$$

The arena for this formula is given in Figure 4. Note that we have also labelled all O -nodes with O and all P -nodes with P .

Next, we define possible moves in our games. Each move for a game for ϕ is associated with a node in the arena for ϕ .

There are several types of moves. Firstly, we have moves by Proponent and Opponent, and secondly there are question and answer moves. Questions which correspond to O -(P -)nodes are played by Opponent (Proponent), and answers which correspond to O -(P -)nodes are played by Proponent (Opponent). The definition is as follows:

Definition 22 A move m for an arena \mathcal{A} is a node which is classified as either question or answer. Questions which correspond to O -(P -)nodes are moves by Opponent (Proponent), and answers which correspond to O -(P -)nodes are moves by Proponent (Opponent). We call a move by Proponent a P -move and a move by Opponent an O -move.


 Figure 4: Arena for $p \supset (p \supset q) \supset (q \supset r) \supset (r \vee s)$

Next, we define plays, which are instances of the game. Each play consists of a sequence of moves satisfying certain conditions. The intuition is that Opponent starts the play by challenging Proponent to verify the given formula. Proponent responds by asking the Opponent to justify the assumptions which Proponent can make in a sequent calculus proof of ϕ . Conjunctive choices are made by Opponent, and disjunctive choices by Proponent. Proponent wins a particular game if he can answer Opponent's initial question.

The moves in a play for ϕ follow the structure of arena of ϕ closely: A O (P)-question can be played only if there was already a P -(O)-question corresponding to the parent node. An answer can only be given if a question with the same associated node has already been made.

The precise conditions for a play are as follows:

Definition 23 A play for an arena \mathcal{A} is a sequence of moves m_1, \dots, m_n such that:

- (i) There exists an index $I \geq 1$ such that all moves m_1, \dots, m_I are O -questions with position $1, \dots, I$ respectively, and the corresponding nodes are roots in the forest for \mathcal{A} . These moves are called initial questions;
- (ii) For each question m_i , with $i > I$, there exists a question m_k , with $k < i$, such that the node corresponding to m_k is the immediate predecessor of the node corresponding to m_i in the arena \mathcal{A} . We call m_k the justifying question for m_i ;
- (iii) For each answer m_i , with $i > I$, there exists a question m_k , with $k < i$, such that m_k and m_i are the same node in \mathcal{A} . If m_j is the justifying question for m_k , we call m_j the justifying question for m_i ;
- (iv) Each question can be answered at most once;
- (v) Any initial questions can only be answered if all non-initial questions have already been answered;
- (vi) For any P -answer m_i there exists a move m_j such that m_j is an O -answer with the same label or \perp and $j < i$ and that the nodes corresponding to m_i and m_j in the arena are on a path which does not contain a P -node n labelled \supset such that the nodes corresponding to m_i and m_j are its children or identical to it;
- (vii) If m is an O -question labelled \vee , then at most one P -question is justified by m .

Condition (vi) of this definition merits an explanation. During plays we have to ensure that Proponent can answer questions of Opponent only if this answer corresponds to an assumption which Opponent has provided. This matters in the case of Proponent asking a question labelled \supset , which corresponds to using an assumption of type $\phi \supset \psi$. The rules of the game work in such a way that in this case two proofs are constructed: one of the original formula using ψ as an additional assumption, and the second one of ϕ . Now we need to ensure that ψ is not available as an assumption during the proof of ϕ . Condition (vi) ensures this by making sure that any O-answer for ϕ cannot be used by Proponent.

Conditions (vii) and (vi) ensure that these games capture intuitionistic proofs: condition (vii) enforces the disjunction property of intuitionistic logic, and condition (vi) makes sure that only one specific formula can be proved at any one given time.

A possible play for the arena for

$$p \supset (p \supset q) \supset (q \supset r) \supset (r \vee s)$$

starts by Opponent asking the initial question. Here, this means that Opponent is asking for a proof of the formula. Now Proponent has various choices: he can either ask questions labelled L or R , thereby deciding whether to prove r or s respectively, or to ask Opponent for evidence for the assumptions by asking any other question. Let us assume that Proponent asks the question corresponding to the node labelled L . Now Opponent will ask the question labelled r , thereby asking Proponent to prove r . Proponent now needs to use the assumptions. Let us assume that Proponent asks the question labelled r , thereby challenging Opponent to provide evidence for the assumption $q \supset r$. Next, Opponent asks the question labelled q and challenges Proponent to prove the formula r in turn, which is the hypothesis in the implication $q \supset r$. Proponent now asks in a similar way the question labelled q , and Opponent asks the question p . Proponent now asks for the final assumption p . Opponent now has no choice but to answer this question, thereby making it possible for Proponent to answer outstanding questions by Opponent. Now Proponent can use this answer and answer Opponent's question p . Again, Opponent is now forced to answer the question q . This process of answering previously asked questions goes on until finally Opponent is forced to answer the question labelled L , and Proponent can answer the initial question. In this example the condition on paths in clause (vi) is not relevant.

The key notion of games semantics is that of a *strategy*. A strategy describes how Proponent responds to arbitrary Opponent moves. Intuitively, a strategy describes how Proponent answers challenges from Opponent to prove the given formula.

Definition 24 A strategy is a function from plays m_1, \dots, m_k , where m_k is an O-move, to a sequence of moves m_{k+1}, \dots, m_n such that $m_1, \dots, m_k, m_{k+1}, \dots, m_n$ is a play, and the sequence m_{k+1}, \dots, m_n is non-empty if the sequence m_1, \dots, m_k contains no unanswered P-move which could be answered by Opponent in the next move according to Definition 23.

Note that this definition makes it possible to force Opponent to answer any unanswered questions by Proponent if such a move was allowed by choosing the empty sequence as a result of the function for sequences with unanswered questions by Proponent.

Intuitively, O- and P-questions are challenges for Opponent and Proponent to provide evidence for conclusions and assumptions respectively. O-answers provide evidence for an assumption, and P-answers provide evidence for a conclusion.

In the example, a strategy for Proponent would be to answer the initial question by asking the question labelled L and then play as indicated above in response to any Opponent move. Note that the choice of asking the question labelled R will not lead to a winning play: Proponent will be unable to answer Opponent's question s .

Next we show that each strategy for the arena corresponding to a formula ϕ gives rise to a sequent calculus proof of ϕ . Note that several strategies give rise to the same proof: games make significantly finer distinctions than sequent calculus proofs.

Theorem 25 *For any formula ϕ and strategy Φ for ϕ there exists a sequent calculus proof of ϕ .*

We have to show a stronger version of this theorem, namely the following version:

Lemma 26 *Given any set A of O -answers with labels p_1, \dots, p_n and a strategy for a formula ϕ where Proponent can answer in addition any O -question with label p_1, \dots, p_n there is a sequent calculus proof of*

$$p_1 \wedge \dots \wedge p_n \supset \phi.$$

We call such a strategy an A -strategy.

Proof By induction over the structure of ϕ . Let Γ be the formula $p_1 \wedge \dots \wedge p_n$.

Atom p : All possible strategies start with a p -question by Opponent. If p is amongst the labels p_1, \dots, p_n of A , then the axiom $p \vdash p$ followed by a \supset -introduction rule provides the desired derivation. There is no other strategy for such an arena;

$\phi \wedge \psi$: Because every question and answer of a strategy for ϕ and ψ has to be justified eventually by an initial move for ϕ and ψ it is possible to obtain one strategy for ϕ and one strategy for ψ from the given strategy. Hence by the induction hypothesis we obtain sequent calculus proofs of $\Gamma \supset \phi$ and $\Gamma \supset \psi$. Hence one obtains also a sequent calculus proof of $\Gamma \supset (\phi \wedge \psi)$;

$\phi \supset \psi$: There are several subcases.

Firstly, suppose $\phi = \phi_1 \wedge \phi_2$. Then $(\phi_1 \wedge \phi_2) \supset \psi$ is equivalent to $\phi_1 \supset \phi_2 \supset \psi$, and the arenas for

$$(\phi_1 \wedge \phi_2) \supset \psi \quad \text{and} \quad \phi_1 \supset \phi_2 \supset \psi$$

are identical. Hence we consider the case $\phi_1 \supset \phi_2 \supset \psi$ instead.

Secondly, suppose $\phi = \sigma \vee \tau$. Now define two A -strategies Φ_1 and Φ_2 for $\sigma \supset \psi$ and $\tau \supset \psi$ respectively, where the moves of both players in Φ_1 and Φ_2 are the moves of Φ which are justified by moves not hereditarily justified by τ or σ respectively. By considering an Opponent-strategy which does not ask the nodes marked L or R corresponding to the disjunction in $\sigma \vee \tau$, one can show that the A -strategies Φ_1 and Φ_2 are well-defined. By the induction hypothesis, we obtain sequent calculus proofs of $\Gamma \supset (\sigma \supset \psi)$ and $\Gamma \supset (\tau \supset \psi)$. Hence there is also a sequent calculus proof of $\Gamma \wedge (\sigma \vee \tau) \supset \psi$.

Thirdly, suppose $\phi = \sigma \supset \tau$. Again, define A -strategies Φ_1 for $\tau \supset \psi$ and Φ_2 for σ where the moves of both players are the ones not hereditarily justified by σ or τ respectively. Clause 6 of definition 23 ensure that these A -strategies Φ_1 and Φ_2 are well-defined. By the induction hypothesis we obtain sequent calculus proofs of $\Gamma \supset \tau \supset \psi$ and $\Gamma \supset \sigma$. Hence there is also a sequent calculus proof of $\Gamma \supset (\sigma \supset \tau) \supset \psi$.

Finally, suppose that ϕ is an atom p . Again, there are two cases. Consider an A -strategy for $p \supset \psi$ without a Proponent-question corresponding to p . In this case, the A -strategy for $p \supset \psi$ is in fact a strategy for ψ , and by the induction hypothesis there is a sequent calculus proof of $\Gamma \supset \psi$, hence also a proof of $\Gamma \supset p \supset \psi$. Now suppose there is a Proponent-question corresponding to p . Clause 6 of Definition 23 ensures that the strategy which removes the Proponent-question and O -answer for p is a $A \cup \{p\}$ -strategy of ψ . By the induction hypothesis there is a sequent calculus proof of $(\Gamma \wedge p) \supset \psi$.

$\phi \vee \psi$: By Clause 7 of Definition 23, an A -strategy for $\phi \vee \psi$ gives rise either to an A -strategy for ϕ or an A -strategy for ψ , depending whether Proponent plays the L -or the R -node. By the induction hypothesis there is either a sequent calculus proof of $\Gamma \vdash \phi$ or of $\Gamma \vdash \psi$. Hence in both cases there is also a sequent calculus proof of $\Gamma \vdash \phi \vee \psi$.

□

We now describe how to extend our games model of intuitionistic proofs to be a model of intuitionistic reductions. For this, we need additional structure to model indeterminates. The key idea is to introduce additional plays which Proponent may start at will.

Definition 27 *A strategy with oracle of type ϕ is a strategy where, in addition, Proponent is allowed to play using an additional arena for ϕ . The justifying question for the root nodes of ϕ is an initial question.*

Substitution of reductions for indeterminates is modelled by substitution of strategies for oracles.

Definition 28 *Suppose Ψ is a strategy with oracle of type ϕ and Φ is a strategy of type ϕ . We define the substitution of Φ for the oracle in Ψ to be the strategy Ψ except that we replace every answer which is a move given by the arena for ϕ by the move obtained by using Φ to answer Ψ 's move in ϕ , then using Ψ to answer this move and so on until Ψ answers with a move outside the arena for ϕ .*

Substitution of strategies for oracles is well-defined:

Lemma 29 *Let Ψ be a strategy for the arena for σ with oracle of type ψ and Φ is strategy for the arena for ψ with an oracle of type ϕ . Then the substitution of Φ for the oracle in Ψ is a strategy for the arena for σ with oracle of type ϕ .*

Proof By induction over the structure of σ . □

A proof with indeterminate of type ϕ is now modelled as a strategy with oracle of type Φ . More precisely, the proof of $\text{?- } \phi$ using only an indeterminate of type ϕ is modelled by the copy-cat strategy, where Proponent simply replays each Opponent-question in the arena for ϕ in the additional arena for ϕ he may use.

Theorem 25 can be extended to games with indeterminates and stated intuitionistic reductions:

Theorem 30 *For any formula ϕ and strategy Φ for ϕ with oracles of type ψ_1, \dots, ψ_m there exists a intuitionistic reduction of ϕ with indeterminates of types ψ_1, \dots, ψ_m .*

Proof By Theorem 25 we obtain a reduction of $(\psi_1 \wedge \dots \wedge \psi_m) \supset \psi$ and hence also a reduction of $\psi_1 \wedge \dots \wedge \psi_m \text{ ?- } \psi$. Now we use cuts with the reduction $(\dots(\Psi_1, \Psi_2); \wedge R \dots), \Psi_m); \wedge R$ where Ψ_i is the reduction consisting only of an indeterminate of type ψ_i . □

At this point, we have achieved our first objective. We have a class of abstract structures which supports the triangle of Figure 2. The top left-hand corner represents the basic calculus of queries; the top right-hand corner stands formal language of reductions, built using a class of variables corresponding to indeterminates; and the bottom corner is given by the interpretation of reduction models. However, we do not as yet have a declarative, or truth-functional, semantics of *search*. As we have seen, in the setting of reductive logic, such a semantics can be understood in terms of state.

We will now develop a semantics of proof-search for intuitionistic logic by considering the class of intuitionistic reductions to be embedded in the class of classical reductions, using the techniques introduced in [33, 31, 27]. To this end, we extend our semantics of reduction to the formulation of classical logic based on the $\lambda\mu\nu$ -calculus [33, 31, 27].

$\frac{}{\Gamma, \phi \vdash \phi, \Delta} Ax$	$\frac{\Gamma, \phi \vdash \Delta \quad \Gamma \vdash \phi, \Delta}{\Gamma \vdash \Delta} Cut$
$\frac{\Gamma, \phi, \psi, \Gamma' \vdash \Delta}{\Gamma, \psi, \phi, \Gamma' \vdash \Delta} ExchangeL$	$\frac{\Gamma \vdash \Delta, \phi, \psi, \Delta'}{\Gamma \vdash \Delta, \psi, \phi, \Delta'} ExchangeR$
$\frac{}{\Gamma, \perp \vdash \Delta} \perp L$	$\frac{}{\Gamma \vdash \top, \Delta} \top R$
$\frac{\Gamma, \psi, \psi' \vdash \Delta}{\Gamma, \psi \wedge \psi' \vdash \Delta} \wedge L$	$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma \vdash \phi', \Delta}{\Gamma \vdash \phi \wedge \phi', \Delta} \wedge R$
$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \supset \psi \vdash \Delta} \supset L$	$\frac{\Gamma, \phi \vdash \psi, \Delta}{\Gamma \vdash \phi \supset \psi, \Delta} \supset R$
$\frac{\Gamma, \phi \vdash \Delta \quad \Gamma, \phi' \vdash \psi, \Delta}{\Gamma, \phi \vee \phi' \vdash \psi, \Delta} \vee L$	$\frac{\Gamma \vdash \phi, \phi', \Delta}{\Gamma \vdash \phi \vee \phi', \Delta} \vee R$

Table 2: Classical Sequent Calculus

3 Semantics for Classical Reductive Logic

In this section, we describe a semantics for propositional classical logic viewed as a reductive system. Building on the wealth of proof-theoretic studies of proof-search in classical logic — see, for example, [7, 33, 31, 30, 28, 40] — we take as our point of departure a minor variant of Gentzen’s sequent calculus, LK, given in Table 2. Contraction and Weakening are built into the other rules but, for technical reasons, we include *Exchange*. Note also the absence of the usual rules for negation,

$$\frac{\Gamma \vdash \phi, \Delta}{\Gamma, \neg \phi \vdash \Delta} \neg L \quad \frac{\Gamma, \phi \vdash \Delta}{\Gamma \vdash \neg \phi, \Delta} \neg R.$$

For technical reasons, it is simpler for our semantic purposes to *define* $\neg \phi$ as in the intuitionistic style as $\phi \supset \perp$. In the presence of the classical $\supset R$ rule, $\neg L$ and $\neg R$ are derivable. For convenience, we shall simply refer to this system as LK.

As with the intuitionistic calculus, LJ, the principal virtues of LK’s presentation of intuitionistic proofs as a basis for proof-search are that it admits Cut-elimination and, in the absence of Cut, has the subformula property. Note, however, that the advantages of Cut discussed in § 2 apply equally well to classical logic.

Semantically, we aim to extend the definition of a reduction structure to classical logic, *i.e.*, to LK proofs. To this end, we require a representation of classical proofs for which a non-trivial semantics is available.¹⁶ For this representation, we use Parigot’s $\lambda\mu$ -calculus [24]. We present this calculus and its extension to cover disjunction, and then present a categorical semantics for it.

3.1 The $\lambda\mu\nu$ -calculus

The raw terms of the $\lambda\mu$ -calculus with conjunction are given by the following grammar:

$$t ::= x \mid \lambda x: \phi. t \mid tt \mid \mu \alpha. t \mid [\alpha]t \mid \mu \perp. t \mid [\perp]t \mid \langle t, t \rangle \mid \pi(t) \mid \pi'(t).$$

We assume that the scope of the bracket operator $[\alpha]t$ extends as far to the right as possible, *i.e.*, the term $[\alpha]ts$ is implicitly bracketed as $[\alpha](ts)$. The rules for well-formed $\lambda\mu$ -terms are given in Table 3. The second instances of the rules $[-]$ and μ model contraction and weakening respectively.

¹⁶That is, a semantics which does not identify all proofs of a given a sequent.

$\frac{}{\Gamma, x: \phi \vdash x: \phi, \Delta} Ax$		
$\frac{\Gamma, x: \phi \vdash t: \psi, \Delta}{\Gamma \vdash \lambda x: \phi. t: \phi \rightarrow \psi, \Delta} \rightarrow I$	$\frac{\Gamma \vdash t: \phi \rightarrow \psi, \Delta \quad \Gamma \vdash s: \phi, \Delta}{\Gamma \vdash ts: \psi, \Delta} \rightarrow E$	
$\frac{\Gamma \vdash t: \phi^\alpha, \Delta}{\Gamma \vdash \mu\alpha.t: \phi, \Delta} \mu$	$\frac{\Gamma \vdash t: \phi, \Delta}{\Gamma \vdash [\alpha]t: \phi^\alpha, \Delta} [-]$	
$\frac{\Gamma \vdash t: \Delta}{\Gamma \vdash \mu\alpha.t: \phi, \Delta} \mu$	$\frac{\Gamma \vdash t: \phi, \phi^\alpha, \Delta}{\Gamma \vdash [\alpha]t: \phi^\alpha, \Delta} [-]$	
$\frac{\Gamma \vdash t: \Delta}{\Gamma \vdash \mu\perp.t: \perp, \Delta} \perp I$	$\frac{\Gamma \vdash t: \perp, \Delta}{\Gamma \vdash [\perp]t: \Delta} \perp E$	
$\frac{\Gamma \vdash t: \phi, \Delta \quad \Gamma \vdash s: \psi, \Delta}{\Gamma \vdash \langle t, s \rangle: \phi \wedge \psi, \Delta} \wedge I$	$\frac{\Gamma \vdash t: \phi \wedge \psi, \Delta}{\Gamma \vdash \pi(t): \phi, \Delta} \wedge E$	$\frac{\Gamma \vdash t: \phi \wedge \psi, \Delta}{\Gamma \vdash \pi'(t): \psi, \Delta} \wedge E$

 Table 3: Well-formed $\lambda\mu$ -terms

The definition of the reduction rules requires not only the standard substitution $t[s/x]$, but also a substitution for names $t[s/[\alpha]u]$, which intuitively indicates the term t with all occurrences of a subterm of the form $[\alpha]u$ replaced by s . Again, we need the notion of a term with holes, adapted for the $\lambda\mu$ -calculus. Such a term \mathcal{C} with holes of type ϕ is a $\lambda\mu$ -term which may have also the additional term constructor \perp with the rule $\Gamma \vdash \perp: \phi, \Delta$. The term $\mathcal{C}(u)$ denotes the term \mathcal{C} with the holes textually (with possible variable capture) replaced by u . Then we define $t[\mathcal{C}(u)/[\alpha]u]$, where α is a name and u is a metavariable, by

$$\begin{aligned} x[\mathcal{C}(u)/[\alpha]u] &= x \\ ([\alpha]t)[\mathcal{C}(u)/[\alpha]u] &= \mathcal{C}(t[\mathcal{C}(u)/[\alpha]u]) \end{aligned}$$

and defined on all other expressions by pushing the replacement inside.

Again, there are three kinds of reduction rules: β , η and ζ -rules. The β and η -rules have the same purpose as the β -and η -rules in the simply-typed λ -calculus. The ζ -rules of the $\lambda\mu\nu$ -calculus are variants of the β -rules where the exchange is applied to the right-hand side before a β -rule is applied. This is different from the simply-typed λ -calculus where ζ -rules model permuting reductions over $\vee E$ -rules. The reduction rules are given in Table 4.

The term $[\alpha]t$ realizes the introduction of a name. The term $\mu\alpha.[\beta]t$ realizes the exchange operation: if ϕ^α was part of Δ before the exchange, then ϕ is the principal formula of the succedent after the exchange. Taken together, these terms also provide a notation for the realizers of contractions and weakenings on the right of a multiple-conclusioned calculus. It is also easy to detect whether a formula ψ^β in the right-hand side is, in fact, superfluous, *i.e.*, that there is a derivation of $\Gamma \vdash t: \phi, \Delta'$ in which Δ' does not contain ψ ; it is superfluous if β is not a free name in t . This observation is exploited in the sequel.

The $\lambda\mu$ -calculus has a special formula \perp and treats the formula $\neg\phi$ as $\phi \rightarrow \perp$. The $\perp I$ -and $\perp E$ -rule model the fact that the formula \perp can be freely added to the right-hand side of each derivation. As these two rules suggest, we treat \perp as a special name, and when we have a generic term $\mu\alpha.t$ with $\Gamma \vdash t: \psi, \phi^\alpha, \Delta$, we always include the case $\mu\perp.t$.

Now we turn to the addition of disjunctive types to the $\lambda\mu$ -calculus. We give here only a summary; for a detailed exposition see [33, 31].

β	$(\lambda x:\phi.t)s \rightsquigarrow t[s/x]$
$\zeta \rightarrow$	$(\mu\alpha^{\phi \rightarrow \psi}.t)s \rightsquigarrow \mu\beta^\psi.t[[\beta]us/[\alpha]u]$
	$(\mu\alpha^{\phi \rightarrow \perp}.t)s \rightsquigarrow \mu\perp.t[[\perp]us/[\alpha]u]$
η^μ	$\mu\alpha.[\alpha]s \rightsquigarrow s \text{ if } \alpha \text{ not free in } s$
β^μ	$[\gamma](\mu\alpha.s) \rightsquigarrow s[\gamma/\alpha]$
η^\perp	$\mu\perp.[\perp]t \rightsquigarrow t$
β^\perp	$[\perp]\mu\perp.t \rightsquigarrow t$
ζ^\wedge	$\pi(\mu\alpha^{\phi \wedge \psi}.s) \rightsquigarrow \mu\beta^\phi.s[[\beta]\pi(u)/[\alpha]u]$
	$\pi'(\mu\alpha^{\phi \wedge \psi}.s) \rightsquigarrow \mu\gamma^\psi.s[[\gamma]\pi'(u)/[\alpha]u]$
	$\pi(\mu\alpha^{\perp \wedge \psi}.s) \rightsquigarrow \mu\perp.s[[\perp]\pi(u)/[\alpha]u]$
	$\pi'(\mu\alpha^{\phi \wedge \perp}.s) \rightsquigarrow \mu\perp.s[[\perp]\pi'(u)/[\alpha]u]$
β^\wedge	$\pi(\langle t, s \rangle) \rightsquigarrow t$
	$\pi'(\langle t, s \rangle) \rightsquigarrow s$

 Table 4: Reduction rules of the $\lambda\mu$ -calculus

One possible formulation, with a single minor formula in the premiss is as follows:

$$\frac{\Gamma \longrightarrow \phi_i, \Delta}{\Gamma \longrightarrow \phi_1 + \phi_2, \Delta} \quad i = 1, 2, \quad (1)$$

and yields the usual addition of sums (coproducts) to the realizing λ -terms:

$$t ::= \text{in}_1(t) \mid \text{in}_2(t) \mid \text{case } t \text{ of } \text{in}_1(x) \Rightarrow t \text{ or } \text{in}_2(y) \Rightarrow t$$

An alternative formulation given in [4] which exploits the presence of multiple conclusions:

$$\frac{\Gamma \longrightarrow \phi_1, \phi_2, \Delta}{\Gamma \longrightarrow \phi_1 \vee \phi_2, \Delta}. \quad (2)$$

Later, in § 4, we shall see that this formulation is the more desirable as basis to model reduction operators for proof-search because it maintains a *local representation* of the *global choice* between ϕ_1 and ϕ_2 : Given a local representation, we can hope to avoid backtracking to this point in the search space.

For the $\lambda\mu$ -calculus, however, this latter formulation presents a new difficulty. Suppose the $\lambda\mu$ -sequent $\Gamma \vdash t:\phi, \psi^\beta, \Delta$ is to be the premiss of an application of the $\vee I$ rule. In forming the disjunctive active formula $\phi \vee \psi$, we move the named formula ψ^β from the context to the active position. Consequently, $\vee I$ is formulated as a binding operation on names and we add the following constructs to $\lambda\mu$, to form the grammar of $\lambda\mu\nu$ -terms [33, 31]:

$$t ::= \langle \beta \rangle t \mid \nu\beta.t. \quad (3)$$

The term $\nu\beta.t$ introduces a disjunction and the term $\langle \beta \rangle t$ eliminates one. The associated inference rules are as follows:

$$\begin{array}{c} \frac{\Gamma \vdash t:\phi, \psi^\beta, \Delta}{\Gamma \vdash \nu\beta.t:\phi \vee \psi, \Delta} \vee I \qquad \frac{\Gamma \vdash t:\phi \vee \psi, \Delta}{\Gamma \vdash \langle \beta \rangle t:\phi, \psi^\beta, \Delta} \vee E \\[10pt] \frac{\Gamma \vdash t:\phi, \Delta}{\Gamma \vdash \nu\perp.t:\phi \vee \perp, \Delta} \vee I_\perp \qquad \frac{\Gamma \vdash t:\phi \vee \perp, \Delta}{\Gamma \vdash \langle \perp \rangle t:\phi, \Delta} \vee E_\perp. \end{array}$$

To avoid variable capture, we have to add a special clause for the mixed substitution:

$$(\langle \alpha \rangle t)[[\mathcal{C}(u)/[\alpha]u]] = \mu\gamma.\mathcal{C}(\mu\alpha.[\gamma]\langle \alpha \rangle t[\mathcal{C}(u)/[\alpha]u])$$

where γ is a fresh name. If we had pushed the substitution through, the substitution lemma fails: the term $\mu\beta.[\alpha]\langle \alpha \rangle x$ is well-formed if x is of type $\phi \vee (\psi \supset \chi)$. If the term $(\mu\beta.[\alpha]\langle \alpha \rangle x)[[\alpha']us/[\alpha]u]$ is defined as $\mu\beta.[\alpha'](\langle \alpha \rangle xs)$, we obtain an ill-formed term.

The corresponding reduction rules are

$$\begin{array}{lll} \beta^\vee & \langle \beta \rangle (\nu\alpha.s) & \rightsquigarrow s[\beta/\alpha] \\ \zeta^\vee & \langle \beta \rangle \mu\gamma.t & \rightsquigarrow \mu\alpha.t[[\alpha]\langle \beta \rangle s/[\gamma]s] \\ \beta_\perp^\vee & \langle \beta \rangle \mu\gamma^{\perp\vee\psi}.t & \rightsquigarrow \mu\perp.t[[\perp]\langle \beta \rangle s/[\gamma]s] \\ \zeta_\perp^\vee & \langle \perp \rangle \mu\gamma^{\phi\vee\perp}.t & \rightsquigarrow \mu\alpha.t[[\alpha]\langle \perp \rangle s/[\gamma]s]. \end{array}$$

The rules $\vee I_\perp$, $\vee E_\perp$, β_\perp^\vee and ζ_\perp^\vee are special cases of $\vee I$, $\vee E$, β^\vee and ζ^\vee , respectively. They are included as convenient abbreviations and need not be analysed separately.

Remark To avoid loops during reduction, all ζ -rules do not apply if the term t in which the name α is changed is equal to $\langle \alpha \rangle t'$, and α does not occur in t' .

Parigot gives only reduction rules for β -reduction. For both proof-theoretic and semantic reasons, we also need *extensionality*, i.e., we must have the η -rules. We will work with long η -normal forms in the sequel.

We introduce them here as expansions; that is, each term of functional type is transformed into a λ -abstraction, each term of product type into a product and each term of sum type into a term $\nu\beta.t'$. These rules are

$$\begin{array}{lll} \eta^\rightarrow & t & \rightsquigarrow \lambda x:\phi.tx \\ \eta^\wedge & t & \rightsquigarrow \langle \pi(t), \pi'(t) \rangle \\ \eta^\vee & t & \rightsquigarrow \nu\alpha.\langle \alpha \rangle t \end{array}$$

In these rules, we assume that t is neither a λ -abstraction, nor a product, nor a term $\nu\alpha.t'$, nor that t occur as the first argument of an application, or as the argument of a projection π or π' or of some term $\langle \beta \rangle _$. In the η^\rightarrow -, η^\wedge - and η^\vee -rules, we also assume that t is of function type, product type and sum type respectively.

These η -rules generate *critical pairs*¹⁷ which give rise to additional reduction rules. As an example, consider the term $t = [\alpha]\mu\alpha.s$, where α is a name of type $\phi \rightarrow \psi$. This term can reduce via an η -expansion to $[\alpha]\lambda x:\phi.(\mu\alpha.t)x$, and via a $\mu\nu$ -rule to t . The reduction from $[\alpha]\lambda x:\phi.(\mu\alpha.t)x$ to t can be seen as a generalized renaming operation. This operation is denoted by $t\{\beta\}$ and is defined as follows:

Definition 31 Define the generalized renaming of a $\lambda\mu\nu$ -term t by a name β , written $t\{\beta\}$, by induction over the type of the name β as follows:

Atomic type: $(\mu\alpha.t)\{\beta\} = t[\beta/\alpha]$;

$\phi \rightarrow \psi$: $(\lambda x:\phi.t)\{\beta\} = t\{\beta'\}[[\beta]\lambda x:\phi.u/[\beta']u]$ for some fresh name β' if x occurs in $t\{\beta'\}$ only within the scope of $[\beta']u$, otherwise $(\lambda x:\phi.t)\{\beta\}$ is undefined;

$\phi \wedge \psi$: If $t = \langle t_1, t_2 \rangle$ and for some names β_1 and β_2 of type ϕ and ψ respectively, $t_2\{\beta_2\}$ arises from $t_1\{\beta_1\}$ by replacing each subterm $[\beta_1]s_1$ recursively by some subterm $[\beta_2]s_2$, then $t\{\beta\} = t_1\{\beta_1\}[[\beta]\langle s_1, s_2 \rangle/[\beta_1]s_1]$;

¹⁷A formal definition of critical pairs may be found in J.W. Klop's comprehensive reference article on term rewriting systems [18]. Informally, the idea is that critical pairs are those pairs of terms upon which the normalization and confluence properties of a rewriting system depend. That is, pairs $\langle \text{angle } t_1, t_2 \rangle$ such that there is a term t such that $t_1 \rightsquigarrow t$ and $t_2 \rightsquigarrow t$.

$\phi \vee \psi$: $(\nu\alpha.t)\{\beta\} = t\{\beta'\}[\beta]\nu\alpha.u/[\beta']u$ for some fresh name β' if α occurs in $t\{\beta'\}$ only within the scope of $[\beta']u$, otherwise $(\nu\alpha.t)\{\beta\}$ is undefined.

The additional reduction rule, which is called ζ^μ , can now be stated as:

$$\zeta^\mu \quad [\alpha]t \rightsquigarrow t\{\alpha\} \quad (4)$$

Note that this reduction rule specializes to the rule β^μ if α is a name of atomic type. Because the outermost bindings $\mu\alpha..$ of names of atomic type disappear by an application of the ζ^μ -rule, this rule cannot give rise to reduction sequences $t \rightsquigarrow^* t$. Logically, the ζ^μ -rule amounts to taking an introduction rule and moving it above a structural rule (*i.e.*, weakening, contraction) applied to its principal formula.

Theorem 32 *The $\lambda\mu\nu$ -calculus is confluent and strongly normalizing.*

Proof See [27]. □

3.2 A Categorical Semantics for the $\lambda\mu\nu$ -calculus

In this subsection we describe the categorical semantics for the $\lambda\mu\nu$ -calculus which we will extend later to obtain a semantics for classical reductions. We give here only the definitions; for the proofs and further background see [27].

We must interpret $\lambda\mu\nu$ -sequents, of the form

$$\Gamma \vdash t : \phi, \Delta.$$

Such a sequent represents, as the term t via the propositions-as-types correspondence [24], a proof of the classical sequent $\Gamma \vdash \phi, \Delta$, in which we forget variables and names. Now, sequents $\Gamma \vdash t : \phi$, which represent, via the propositions-as-types correspondence [9], proofs in intuitionistic propositional logic, can be interpreted in a bi-Cartesian closed category [20]. However, it is well-known that any attempt to extend this interpretation to classical sequents by adding an involutive negation must fail because bi-CCCs with involutions collapse to Boolean algebras, thereby causing the interpretation to identify all proofs of a given sequent. The solution adopted in Ong-Ritter models [23] is to use a fibration, as follows:

- The base \mathcal{B} , which is a category with finite products, interprets the named part of the sequent, Δ . Its arrows $f : \llbracket \Delta \rrbracket \longrightarrow \llbracket \Delta' \rrbracket$ interpret compositions of weakenings, contractions and permutations;
- The fibre $\mathcal{E}_{\llbracket \Delta \rrbracket}$ over each object $\llbracket \Delta \rrbracket$ of the base is Cartesian closed. It interprets sequents of the form $\Gamma \vdash \phi$, with side-formulæ Δ ;
- Finally, we must add sufficient structure to interpret the structural operations, including negation. In particular, we must be able to interpret the exchange rule

$$\frac{\Gamma \vdash t : \psi, \phi^\alpha, \Delta}{\Gamma \vdash \mu\alpha.[\beta]t : \phi, \psi^\beta, \Delta},$$

described in § 3.1. The key point here is that we move from the fibre over ϕ^α, Δ to the fibre over ψ^β, Δ and must have sufficient structure in the fibration, corresponding to the interpretation of μ and $[-]$, to interpret this swap.

It follows that the appropriate categorical definitions of models of $\lambda\mu$, $\lambda\mu\oplus$ and $\lambda\mu\nu$ are as fibrations with universally-defined extra structure corresponding, respectively, to each additional logical connective, \oplus or \vee .

Such models, because they are fibrations, require Beck-Chevalley conditions [36, 15] for each connective which is to be interpreted. These conditions interpret the ζ -rules for the corresponding type-constructors, ensuring the interpretation of the connectives is stable with respect to change of base (*cf.* the use of Beck-Chevalley conditions to ensure that substitution is modelled correctly in hyperdoctrines). The requisite definitions follow.

Definition 33 A $\lambda\mu$ -structure is a split fibration $p: \mathcal{E} \rightarrow \mathcal{B}$ satisfying the following conditions:

1. $p: \mathcal{E} \rightarrow \mathcal{B}$ is a fibred Cartesian closed category, i.e., each fibre is Cartesian closed and re-indexing, i.e., applications of functors f^* , preserves products and function spaces on the nose;
2. The fibre \mathcal{E}_1 over the terminal object 1 in \mathcal{B} is canonical: i.e., for any object D of \mathcal{B} , there is a bijection between the objects of \mathcal{E}_D and \mathcal{E}_1 , with one direction given by re-indexing along the terminal arrow $!_D: D \rightarrow 1$, i.e., applications of the functor $!_D^*$;
3. The base category \mathcal{B} is the free category with finite products generated from the set of objects of the canonical fibre \mathcal{E}_1 less a distinguished object \perp and all objects isomorphic to it (note that all arrows in \mathcal{B} , a free category with finite products, are compositions of weakening, contractions and permutations);

4. For each projection

$$w_A: D \times A \rightarrow D,$$

in the base, there is an isomorphism

$$\mathcal{E}_D(C, A) \cong \mathcal{E}_{D \times A}(\alpha^*(C), \perp),$$

written as $s \xrightarrow{[-]} [\alpha^A]s$ and $\mu\alpha^A . t \xleftarrow{\mu} t$, natural in C and D ;

5. For any object A of a category \mathcal{C} with finite products, the flat fibre \mathcal{C}^A is the category whose objects are objects of \mathcal{C} and the morphisms from B to C are morphisms from $B \times A$ to C . The previous conditions imply the existence of a bijection $\zeta: \mathcal{E}_{\Delta \times A \rightarrow B}^\Gamma(C, D) \xrightarrow{\sim} \mathcal{E}_{\Delta \times B}^{\Gamma \times A}(C, D)$. We require the action ζ to be functorial, natural in Γ and Δ , and to make the following diagram commute:

$$\begin{array}{ccc} \mathcal{E}_{\Delta \times A \rightarrow B}^\Gamma(C, D) & \xrightarrow{\zeta_{\Delta; A, B}} & \mathcal{E}_{\Delta \times B}^{\Gamma \times A}(C, D) \\ \downarrow [\delta^D] & & \downarrow [\delta^D] \\ \mathcal{E}_{\Delta \times A \rightarrow B \times D}^\Gamma(C, \perp) & \xrightarrow{\zeta_{\Delta \times D; A, B}} & \mathcal{E}_{\Delta \times B \times D}^{\Gamma \times A}(C, \perp) \end{array}$$

6. A Beck-Chevalley condition holds for \Rightarrow : for each contraction map

$$c: \Delta \times (A \Rightarrow B) \rightarrow \Delta \times (A \Rightarrow B) \times (A \Rightarrow B)$$

in \mathcal{B} we require the following diagram to commute:

$$\begin{array}{ccc} \mathcal{E}_{\Delta \times A \Rightarrow B \times A \Rightarrow B}^\Gamma & \xrightarrow{\zeta \circ \zeta} & \mathcal{E}_{\Delta \times B \times B}^{\Gamma \times A \times A} \\ \downarrow c_{A \Rightarrow B}^* & & \downarrow c_B^* \circ c_A^* \\ \mathcal{E}_{\Delta \times A \Rightarrow B}^\Gamma & \xrightarrow{\zeta} & \mathcal{E}_{\Delta \times B}^{\Gamma \times A} \end{array}$$

Note that in the composite arrow $c_B^* \cdot c_A^*$, and subsequent similar situations, we overload our notation (as in [23]) by writing c_A^* for re-indexing along the relevant “contraction map” in the flat fibration over $\mathcal{E}_{\Delta \times B \times B}$;

7. A Beck-Chevalley condition holds for products: for the canonical isomorphism and the contraction functor, namely

$$\phi: \mathcal{E}_{\Delta \times (A \times B)} \rightarrow \mathcal{E}_{\Delta \times A} \times \mathcal{E}_{\Delta \times B} \quad \text{and} \quad c_A: \mathcal{E}_{\Delta \times A \times A} \rightarrow \mathcal{E}_{\Delta \times A},$$

the two functors

$$(c_A^* \times \langle \rangle \times \langle \rangle \times c_B^*) \circ (\phi \times \phi) \circ \phi: \mathcal{E}_{\Delta \times A \times B \times A \times B} \rightarrow \mathcal{E}_{\Delta \times A} \times \mathcal{E}_{\Delta \times B}$$

and

$$\phi \circ c_{A \times B}^*: \mathcal{E}_{\Delta \times A \times B \times A \times B} \rightarrow \mathcal{E}_{\Delta \times A} \times \mathcal{E}_{\Delta \times B}$$

are equal.

Definition 34 A $\lambda\mu$ -model is a pair $\mathcal{P} = \langle p, [-] \rangle$, where $p: \mathcal{E} \rightarrow \mathcal{B}$ is a $\lambda\mu$ -structure and the interpretation $[-]: L_{\lambda\mu} \rightarrow p$ is a function from the syntax of $\lambda\mu$ (denoted $L_{\lambda\mu}$) to (the components of) p such that $[\Delta]$ is an object of \mathcal{B} and $\Gamma \vdash t: A, \Delta$ is interpreted as morphism $[[t]]: [\Gamma] \rightarrow [A]$ in the fibre over $[\Delta]$. The interpretations of variables, pairs and λ -abstractions are given in the usual way via projections, products and the exponentials in the fibres, respectively. The terms $\mu\alpha.t$ and $[\alpha]t$ are interpreted by the isomorphism given in Definition 33 (4).

We will sometimes write $\mathcal{E}_{\mathcal{P}}(D)$ for the fibre over D in the model \mathcal{P} and $\mathcal{B}_{\mathcal{P}}$ for the base in the model \mathcal{P} . Also, we will sometimes write $[-]_{\mathcal{P}}$ to denote interpretation in the model \mathcal{P} . We extend structures to account for each of the two forms of disjunction in the next two definitions. In each case, the corresponding definition of model requires an interpretation $[-]$, extended to $L_{\lambda\mu\oplus}$ and $L_{\lambda\mu\nu}$, respectively, as in Definition 34.

Definition 35 A $\lambda\mu$ -structure is called a $\lambda\mu\oplus$ -structure if each fibre has a coproduct which is stable under re-indexing, i.e., applications of the functor f^* , where f is any morphism of \mathcal{B} . Additionally, we require the following Beck-Chevalley condition: the diagram

$$\begin{array}{ccc} \mathcal{E}_{\Delta \times C}^{\Gamma \times (A+B)}(w_C^* E, \perp) & \xrightarrow{\mu} & \mathcal{E}_{\Delta}^{\Gamma \times (A+B)}(E, C) \\ \downarrow \iota_{(A+B)} & & \downarrow \iota_{(A+B)} \\ \mathcal{E}_{\Delta \times C}^{\Gamma \times A}(w_C^* E, \perp) \times \mathcal{E}_{\Delta \times C}^{\Gamma \times B}(w_C^* E, \perp) & \xrightarrow{\mu \times \mu} & \mathcal{E}_{\Delta}^{\Gamma \times A}(E, C) \times \mathcal{E}_{\Delta}^{\Gamma \times B}(E, C) \end{array}$$

commutes, where $\iota_{(A+B)}$ is the defining isomorphism for the co-product in the fibres. The definition of interpretation $[-]$ can be adapted to $\lambda\mu\oplus$ -structures in order to give $\lambda\mu\oplus$ -models as follows: the term constructors **case**, **in**₁ and **in**₂ are interpreted by the corresponding co-product constructions.

Given this definition of $\lambda\mu\oplus$ -models, we can establish soundness and completeness for $\lambda\mu\oplus$ quite straightforwardly.

Definition 36 A $\lambda\mu$ -structure is called a $\lambda\mu\nu$ -structure if each weakening functor $w_{\Delta, A}^*: \mathcal{E}_{\Delta} \rightarrow \mathcal{E}_{\Delta \times A}$ has a right adjoint. We denote by ν the defining isomorphism

$$\nu: \text{hom}_{\mathcal{E}(\Delta \times B)}(\Gamma, A) \xrightarrow{\sim} \text{hom}_{\mathcal{E}(\Delta)}(\Gamma, A \vee B).$$

We also ask for this adjunction to satisfy a Beck-Chevalley condition, i.e., that the diagram

$$\begin{array}{ccc} \mathcal{E}_{\Delta \times A \vee B \times A \vee B} & \xrightarrow{\zeta_{\vee} \circ \zeta_{\vee}} & \mathcal{E}_{\Delta \times A \times B \times A \times B} \\ \downarrow c^* & & \downarrow c^* \circ c^* \\ \mathcal{E}_{\Delta \times A \vee B} & \xrightarrow{\zeta_{\vee}} & \mathcal{E}_{\Delta \times A \times B} \end{array}$$

commutes, where ζ_V is the functor given by assigning each morphism $f: C \rightarrow D$ in $\mathcal{E}_{\Delta \times A \vee B}$ the morphism $\mu\gamma.[\alpha](\nu^{-1}(\mu\beta.[\gamma]f))$. The definition of interpretation $\llbracket - \rrbracket$ can be adapted to $\lambda\mu\nu$ -structures in order to give $\lambda\mu\nu$ -models as follows: the interpretation of terms $\nu\alpha.t$ and $\langle\alpha\rangle t$ uses the defining isomorphism for \vee .

$\lambda\mu\nu$ -structures are sound and complete for the $\lambda\mu\nu$ -calculus (see [33, 31] for details).

Next we adapt the semantics of LK-proofs in the $\lambda\mu\nu$ -calculus to deal with LK-reductions, in the same way as we changed the semantics of LJ-proofs using bi-Cartesian closed categories to deal with LJ-reductions.

3.3 Classical Reduction Models

Having established the semantics of $\lambda\mu\nu$ as a deductive system, and given our general prescription for reading inference rules as reduction operators, we can give the definition of a classical reduction structure. Such a structure arises from a $\lambda\mu\nu$ -structure by introducing an additional fibration to model indeterminates and introducing a category of worlds and a functor to the Grothendieck-completion of the fibration as for reduction structures. Note that we can merge the two fibrations (one for the formulæ on the right hand side, and one for indeterminates) into a fibration over a product.

Again, a few points are noteworthy:

- The addition of indeterminates to models of $\lambda\mu\nu$ follows the same pattern as for (intuitionistic) reduction structures but fibre-wise;
- The structure of $\lambda\mu\nu$ -models reflects the fact that $\lambda\mu\nu$ is essentially a system of natural deduction. Consequently, just as in the intuitionistic case, the interpretation of (Cut-free) LK reductions exploits a Kripke-world structure which records the history of the reduction;
- As before, there is no equality between reductions in the semantics: We interpret only Cut-free reductions and do not consider any equality induced by Cut-elimination. A non-trivial, symmetric categorical semantics of LK (essentially in Gentzen's original form [8], which validates all (in)equalities induced by Cut-elimination, has been introduced by Führmann and Pym [6], but these ideas are beyond our present scope.

Definition 37 Let \mathcal{W} be a small category (of ‘worlds’) with finite products. A classical reduction structure (\mathcal{E}, F) is given by the following:

- (i) A strict indexed category $\mathcal{E}: (\mathcal{B} \times \mathcal{C})^{op} \rightarrow \mathbf{Cat}$ with comprehension such that \mathcal{B} has finite products and each fibre $\mathcal{E}(\Gamma, \Delta)$ is a bi-Cartesian closed category and each functor $\mathcal{E}(f, g)$ preserves the bi-Cartesian closed structure on the nose; and
- (ii) A functor $F: \mathcal{W} \rightarrow \mathcal{B}$ which preserves finite products;

such that the following properties hold:

- (i) There is a natural bijection between $\mathrm{Hom}_{\mathcal{B}}(A, B \times C)$ and the pair

$$(\mathrm{Hom}_{\mathcal{B}}(A, B), \mathrm{Hom}_{\mathcal{E}(A, 1)}(1, C));$$

- (ii) For each object A of \mathcal{B} , the functor $\mathcal{E}(\mathrm{Id}_A \times -)$ is $\lambda\mu\nu$ -structure, and for each morphism $f: A \rightarrow B$ in \mathcal{B} , the natural transformation $\mathcal{E}(f, -)$ preserves the structure of a $\lambda\mu\nu$ -structure on the nose.

We will give a set-theoretic example of a classical reduction structure using continuations in § 5.

Next we describe how to interpret LK-reductions in a classical reduction structure. In the same way as for intuitionistic logic, we first spell out the defining conditions for such an interpretation.

Definition 38 (interpretation) Let (\mathcal{E}, F) be a classical reduction structure. A function $\llbracket - \rrbracket$, which is parametrized by a list of indeterminates Θ and a world W , mapping reductions of LK and their syntactic constituents to elements of a reduction structure is called an interpretation if it satisfies the following mutually recursive conditions:

- (i) $\llbracket \Theta \rrbracket^W$ is an object of \mathcal{B} and $\llbracket \Theta \rrbracket^W = A$ if Θ is the empty list of indeterminates and $F(W) = A$;
- (ii) For any formula ϕ , $\llbracket \phi \rrbracket_\Theta^W$ is an object of the category $\mathcal{E}(\llbracket \Theta \rrbracket^W, 1)$;
- (iii) For any context $\Gamma = \phi_1, \dots, \phi_n$, $\llbracket \Gamma \rrbracket_\Theta^W$ is equal to $(A_1 \times \dots \times A_n)$, where $\llbracket \phi_i \rrbracket_\Theta^W = A_i$;
- (iv) For a reduction $\Phi; \Gamma \text{ ?- } \phi, \Delta$ with indeterminates in Θ , $\llbracket \Phi \rrbracket_\Theta^W$ is a pair (W', g) , where W' is a world and g a morphism from $\llbracket \Gamma \rrbracket_\Theta^W$ to $\llbracket \phi \rrbracket_\Theta^W$ in $\mathcal{E}(\llbracket \Theta \rrbracket^{W'}, \llbracket \Delta \rrbracket_\Theta^{W'})$ such that $g = (\langle \text{Id}, F(a) \rangle, \text{Id})^* f$, for some morphisms

$$f: \llbracket \Gamma \rrbracket_\Theta^{W'} \rightarrow \llbracket \phi \rrbracket_\Theta^{W'}$$

and $a: W \rightarrow W'$;

- (v) For all reduction operators R , there exists a world W_R and a morphism $a_R: 1 \rightarrow W_R$;
- (vi) For a reduction $\Phi; R$, with unary reduction operator R , with sufficient premiss $\Gamma' \text{ ?- } \phi', \Delta'$ and putative conclusion $\Gamma \text{ ?- } \phi, \Delta$,

$$\begin{aligned} \llbracket \Phi; R \rrbracket_\Theta^W &= (W', (\langle \text{Id}, F(a) \rangle, \text{Id})^* (\nu_{\llbracket \Delta \rrbracket_\Theta^{W \times W_R}}^{-1} (\text{Cur}_{\llbracket \Gamma \rrbracket_\Theta^{W \times W_R}}^{-1} \\ &\quad (\text{App} \circ \langle \text{Cur}_{\llbracket \Gamma_1 \rrbracket_\Theta^{W \times W_R}} (\nu_{\llbracket \Delta' \rrbracket_\Theta^{W \times W_R}} (f_1)), \text{Snd})))))) \end{aligned}$$

where $W' = W_1 \times W \times W_R$ and $\llbracket \Phi \rrbracket_\Theta^{W \times W_R} = (W', f_1)$ and $a: W \rightarrow W'$;

- (vii) For a reduction $(\Phi_1, \Phi_2); R$, with binary reduction operator R , with sufficient premisses $\Gamma_i \text{ ?- } \phi_i, \Delta_i$ and with putative conclusion $\Gamma \text{ ?- } A, \Delta$,

$$\begin{aligned} \llbracket (\Phi_1, \Phi_2); R \rrbracket_\Theta^W &= (W', (\langle \text{Id}, F(a) \rangle, \text{Id})^* (\nu_{\llbracket \Delta \rrbracket_\Theta^{W \times W_R}}^{-1} \\ &\quad (\text{Cur}_{\llbracket \Gamma \rrbracket_\Theta^{W \times W_R}}^{-1} (\text{App} \circ \langle \text{Cur}_{\llbracket \Gamma_1 \rrbracket_\Theta^{W \times W_R}} (\nu_{\llbracket \Delta_1 \rrbracket_\Theta^{W \times W_R}} (f_1)), \\ &\quad \text{Cur}_{\llbracket \Gamma_1 \rrbracket_\Theta^{W \times W_R}} (\nu_{\llbracket \Delta_2 \rrbracket_\Theta^{W \times W_R}} (f_2))))), \text{Snd}))) , \end{aligned}$$

where $W' = W_1 \times W_2 \times W \times W_R$, $a: W \rightarrow W'$ and $\llbracket \Phi_i \rrbracket_\Theta^{W \times W_R} = (W'_i, f_i)$ and $W'_i = W \times W_R \times W_i$;

- (viii) Suppose $\Theta = \Theta', \xi$, where ξ is an indeterminate $\phi_1, \dots, \phi_n \text{ ?- } \phi$. Then $\llbracket \Theta \rrbracket^W$ is equal to $\llbracket \Theta' \rrbracket^W \times \llbracket (\phi_1 \wedge \dots \wedge \phi_n) \supset \phi \rrbracket_{\Theta'}^W$.

We can now give the canonical interpretation of LK-reductions in classical reduction structures.

Definition 39 (canonical interpretation) Let (\mathcal{E}, F) be a classical reduction structure. The following interpretation, $\llbracket - \rrbracket$, where Θ is a list of indeterminates, is called the canonical interpretation (where ass is the associativity isomorphism between $(\phi \vee \psi) \vee \Delta$ and $\phi \vee (\psi \vee \Delta)$):

- (i) $\llbracket \perp \rrbracket_\Theta^W \stackrel{\text{def}}{=} 0$;
- (ii) $\llbracket \top \rrbracket_\Theta^W \stackrel{\text{def}}{=} 1$;
- (iii) $\llbracket \phi \supset \psi \rrbracket_\Theta^W \stackrel{\text{def}}{=} \llbracket \phi \rrbracket_\Theta^W \Rightarrow \llbracket \psi \rrbracket_\Theta^W$;
- (iv) $\llbracket \phi \wedge \psi \rrbracket_\Theta^W \stackrel{\text{def}}{=} \llbracket \phi \rrbracket_\Theta^W \times \llbracket \psi \rrbracket_\Theta^W$;
- (v) $\llbracket \phi \vee \psi \rrbracket_\Theta^W \stackrel{\text{def}}{=} \llbracket \phi \rrbracket_\Theta^W \vee \llbracket \psi \rrbracket_\Theta^W$;

(vi) For all reduction operators R except $\supset L$, $\vee L$, $\vee R$ and *ExchangeR*, $F(a_R) = \langle \text{Id}_1, f \rangle$, where $S_\xi(f)$ is the interpretation of R according to Definition 4, where the category \mathcal{C} is the category $\mathcal{E}(1, 1)$;

(vii) For the remaining reduction operators, $F(a_R)$ is defined as follows:

ExchangeR Consider the reduction operator

$$\frac{\Gamma \text{ ?- } \psi, \phi, \Delta}{\Gamma \text{ ?- } \phi, \psi, \Delta}$$

and let ϕ' be the formula $\Gamma \supset \psi \vee \phi \vee \Delta$. Then

$$F_a(\text{ExchangeR}) = \langle \text{Id}, \text{Cur}(\nu(\mu\alpha.[\beta]\text{App} \circ \langle \nu^{-1}(\pi_{[\phi']}), \pi_{[\Gamma]} \rangle)) \rangle;$$

$\supset L$: Consider the reduction operator

$$\frac{\Gamma \text{ ?- } \phi, \sigma, \Delta \quad \Gamma, \psi \text{ ?- } \sigma, \Delta}{\Gamma, \phi \supset \psi \text{ ?- } \sigma, \Delta}$$

and let ϕ_1 be the formula $(\Gamma \supset \phi \vee \sigma \vee \Delta)$, ϕ_2 be $(\Gamma \wedge \psi) \supset \sigma \vee \Delta$, and let π_1 be the projection

$$\llbracket \phi_1 \wedge \phi_2 \wedge \Gamma \wedge (\phi \supset \psi) \rrbracket_\emptyset^1 \rightarrow \llbracket \phi_1 \rrbracket_\emptyset^1,$$

π_2 the projection

$$\llbracket \phi_1 \wedge \phi_2 \wedge \Gamma \wedge (\phi \supset \psi) \rrbracket_\emptyset^1 \rightarrow \llbracket \phi_2 \rrbracket_\emptyset^1.$$

Then

$$\begin{aligned} F(a_{\supset L}) = & \langle \text{Id}_1, \text{Cur}(\mu\gamma.c^*([\gamma](\nu^{-1}(\text{Cur}^{-1}(\pi_2))) \circ \\ & \langle \text{Id}, \pi_{[\Gamma]}, \text{App} \circ \langle \pi_{[\phi \supset \psi]_\emptyset^1}, \nu^{-1}(\text{Cur}^{-1}(\pi_1)) \circ \langle \text{Id}, \pi_{[\Gamma]} \rangle \rangle \rangle \rangle) \rangle; \end{aligned}$$

$\vee L$: Consider the reduction operator

$$\frac{\Gamma, \phi \text{ ?- } \sigma, \Delta \quad \Gamma, \psi \text{ ?- } \sigma, \Delta}{\Gamma, \phi \vee \psi \text{ ?- } \sigma, \Delta}$$

and let ϕ_1 be the formula $(\Gamma \wedge \phi) \supset \sigma \vee \Delta$, ϕ_2 be $(\Gamma \wedge \psi) \supset \sigma \vee \Delta$, and let π_1 be the projection

$$\llbracket \phi_1 \wedge \phi_2 \wedge \Gamma \wedge (\phi \vee \psi) \rrbracket_\emptyset^1 \rightarrow \llbracket \phi_1 \rrbracket_\emptyset^1$$

and π_2 be the projection

$$\llbracket \phi_1 \wedge \phi_2 \wedge \Gamma \wedge (\phi \vee \psi) \rrbracket_\emptyset^1 \rightarrow \llbracket \phi_2 \rrbracket_\emptyset^1.$$

Then

$$\begin{aligned} F(a_{\vee L}) = & \langle \text{Id}_1, \text{Cur}(\mu\gamma.c^*([\gamma](w^*\pi_2 \circ \langle \text{Id}, \pi_{[\Gamma]_\emptyset^1}, \mu\beta.[\gamma] \\ & (w^*\text{Cur}^{-1}\pi_1 \circ \langle \text{Id}, \pi_{[\Gamma]_\emptyset^1}, \nu^{-1}(\pi_{[\sigma \vee \tau]_\emptyset^1}) \rangle \rangle \rangle) \rangle) \rangle \end{aligned}$$

$$\vee R: F(a_{\vee R}) = \langle \text{Id}_1, \text{Cur}(\nu(\text{ass} \circ \nu^{-1}(\text{App} \circ \langle \pi_{[\Gamma]_\emptyset^1}, \pi_{[\Gamma \supset A \vee B \vee \Delta]_\emptyset^1} \rangle \rangle)) \rangle.$$

(Because reduction structures are derived from $\lambda\mu\nu$ -structures, in the cases for $\supset L$ and $\vee L$, the formula σ is distinguished in order to define the interpretation.)

Note that also in the classical case the definition of interpretation does not force any two reductions to be equal. The reason is the same as for (intuitionistic) reduction structure: No equality between worlds or morphisms between them is forced by the interpretation.

Note that the semantics of the reduction operators which involve structural rules on the right-hand side or change the side formulæ on the right-hand side involve a change of base \mathcal{C} . This

is obviously true for *ExchangeR*, but also $\supset L$, $\forall L$ and $\forall R$ involve such a change of base: for $\supset L$ and $\forall L$ it is given by a contraction on the right-hand side, and for $\forall L$ by the isomorphism used for modelling \vee .

We can now define classical reduction models, which generalize the (intuitionistic) reduction models established in Definition 10.

Definition 40 (classical reduction model) A classical reduction model,

$$\mathcal{R} = \langle (\mathcal{E}, F), \llbracket - \rrbracket, \models \rangle,$$

is given by the following:

- A classical reduction structure (\mathcal{E}, F) ;
- An interpretation $\llbracket - \rrbracket$ of reduction operators and searches for *LK*;
- A forcing relation $W \models_{\Theta} (\Phi: \phi)[\Gamma; \Delta]$, where W is a world, Θ and Γ, Δ are contexts, ϕ a formula and Φ a reduction with endsequent $\Gamma \text{ ?- } \phi$, Δ with indeterminates contained in Θ , such that

$$\llbracket \Gamma \rrbracket_{\Theta}^W \xrightarrow{\llbracket \Phi \rrbracket_{\Theta}^W} \llbracket \phi, \Delta \rrbracket_{\Theta}^W$$

is a morphism in the reduction structure, and which satisfies the following conditions:

1. If $W \models_{\Theta} (\Phi: \phi)[\Gamma; \Delta]$ and $a: W \rightarrow W'$ is a morphism in \mathcal{W} for some world W' , then also $W' \models_{\Theta} (\Phi: \phi)[\Gamma; \Delta]$;
2. $W \models_{\Theta} (Ax: \phi)[\Gamma, \phi; \Delta]$;
3. $W \models_{\Theta, \xi} (\xi: \phi)[\Gamma; \Delta]$ if ξ is an indeterminate of type $\Gamma \text{ ?- } \phi; \Delta$;
4. If R is a reduction operator with premisses $\Gamma_1 \text{ ?- } \phi_1, \Delta_1$ and $\Gamma_2 \text{ ?- } \phi_2, \Delta_2$ and conclusion $\Gamma \text{ ?- } \phi, \Delta$, then $W \models_{\Theta} ((\Phi_1, \Phi_2); R)[\Gamma, \phi; \Delta]$ if

$$W \times W_R \models_{\Theta} (\Phi_i)[\Gamma_i, \phi_i; \Delta_i];$$

5. If R is a reduction operator with premiss $\Gamma_1 \text{ ?- } \phi_1, \Delta_1$ and conclusion $\Gamma \text{ ?- } \phi, \Delta$, then $W \models_{\Theta} (\Phi_1; R)[\Gamma, \phi; \Delta]$ if $W \times W_R \models_{\Theta} (\Phi_1)[\Gamma_1, \phi_1; \Delta_1]$;

Soundness and completeness carry over from the intuitionistic case.

Theorem 41 (soundness) Consider any classical reduction structure (\mathcal{E}, F) . Suppose Φ is a *LK*-reduction of $\Gamma \text{ ?- } \phi, \Delta$ with indeterminates ξ_1, \dots, ξ_n of type $\Gamma_i \text{ ?- } \phi_i, \Delta_i$. Then $W \models_{\Theta} (\Phi: \phi)[\Gamma; \Delta]$ for any world W , where $\Theta = \{\xi_1, \dots, \xi_n\}$.

Proof The proof is essentially the same as for Theorem 12. □

Again, we write $\Gamma \models_{\Theta} \Phi: \phi; \Delta$ if for all worlds W and all classical reduction models, we have $W \models_{\Theta} (\Phi: \phi)[\Gamma; \Delta]$. Then we have also completeness:

Theorem 42 (completeness) If $\Gamma \models_{\Theta} \Phi: \phi$, then $\Gamma \text{ ?- } \Phi: \phi$.

Proof The term model construction for the intuitionistic case can be extended easily to give a term model for a classical reduction structure. For the category \mathcal{C} choose the free cartesian category over the atomic formulæ, and now follow the intuitionistic case in constructing a term model out of reductions. □

We also obtain completeness with respect to searches:

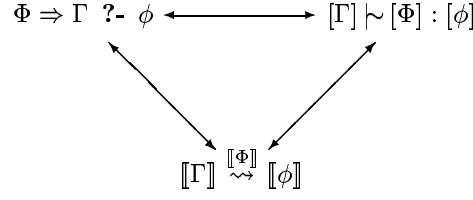


Figure 5: Reductions-as-realizers-as-arrows

Theorem 43 Suppose (\mathcal{E}, F) is the free classical reduction structure over a set of objects \mathcal{G} . A reduction Φ of $\Gamma \text{ ?- } \phi, \Delta$ with leaves $\Gamma_i \text{ ?- } \phi_i, \Delta_i$ which are not axioms can be completed to a proof iff there exists a morphism f such that there is a functor $\mathcal{E}(!, f): \mathcal{E}(\Theta) \rightarrow \mathcal{E}(1)$, where Θ is the context corresponding to the indeterminates ξ_1, \dots, ξ_n . Moreover, the completion of a Cut-free reduction is Cut-free.

Proof The proof can be transferred directly from the intuitionistic case. \square

We have now provided, in the intuitionistic and classical settings, a semantics for reductive proof which satisfies our triangular criterion of Figure 5:

However, we have not yet provided a semantics for proof-search. Following our slogan,

$$\text{Proof-search} = \text{Reductive Proof} + \text{Control},$$

we must now pay attention to control. Hence we shall provide, in § 4, a semantics for backtracking.

We conjecture that the semantics for reductive proof given in this chapter can be easily extended to predicate logic and quantifiers: we have previously described how to use fibrations to obtain models for predicate logic. It should be possible to combine these fibrations in a modular way with the fibrations used to describe reduction structures, so as to produce reduction structures for predicate logic.

4 Towards a Semantics of Control: Backtracking

In this section we provide a semantics for proof-search in intuitionistic (propositional) logic which captures, within the framework of models of reductive logic we have described in § 2 and § 3, backtracking. We achieve this aim firstly, by providing a characterization in classical reduction models of where backtracking can occur in intuitionistic proof-search and, secondly, by constructing a specific games model within which both backtracking and the uniform proof strategies may be understood quite naturally.

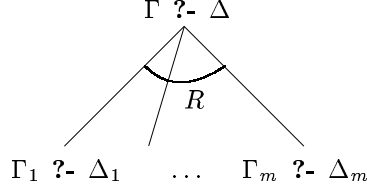
Given a system of reduction operators, \mathcal{R} , the search space of \mathcal{R} , $\text{Space}(\mathcal{R})$, may be described graphically as an *and-or tree* as follows:¹⁸

1. Nodes of the tree are labelled by problems, $\Gamma \text{ ?- } \Delta$. The root is labelled by the initial problem;
2. Nodes are connected by arcs labelled by instances of reduction operators,

$$\frac{\Gamma_1 \text{ ?- } \Delta_1 \quad \dots \quad \Gamma_m \text{ ?- } \Delta_m}{\Gamma \text{ ?- } \Delta} R,$$

¹⁸In [28, 30], the search space for an intuitionistic sequent calculus is defined to carry the ‘subderivation ordering’, \sqsubseteq : For reductions R, S , $R \sqsubseteq S$ if R is a labelled subtree of S . In this paper, we shall make no use of this ordering but remark that orderings of this kind may provide a suitable basis modelling control régimes such as formula-selection strategies. For example, Prolog programs may be seen as antecedents of sequents, ordered from left to right in order to impose the “leftmost first” strategy.

which may be denoted



in which arcs are directed (traditionally) down the page. The collection of arcs from a node labelled by some problem $\Gamma \text{ ?- } \Delta$ to the nodes labelled by the problems

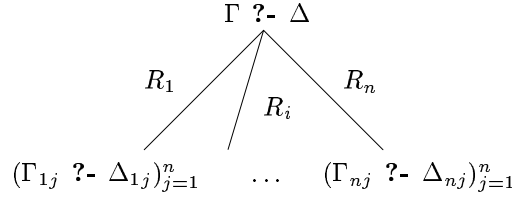
$$\Gamma_1 \text{ ?- } \Delta_1 \dots \Gamma_m \text{ ?- } \Delta_m,$$

determined by such an instance of a reduction operator, R , and connected by the curved arc in the figure, is called an R -bundle;¹⁹

3. A problem may be the origin of several different bundles, corresponding to different reduction operators and giving the disjunctive (or) structure of the space. If n different reduction operators R_i ,

$$\frac{\Gamma_{i1} \text{ ?- } \Delta_{i1} \dots \Gamma_{im} \text{ ?- } \Delta_{im}}{\Gamma \text{ ?- } \Delta} R_i,$$

for $1 \leq i \leq n$, are applicable to a problem $\Gamma \text{ ?- } \Delta$, then the corresponding arcs in the search space may be denoted



i.e., a disjunction of R_i -bundles;

4. Paths through $Space(\mathcal{R})$ thus correspond to compositions of instances of reduction operators.

Within a bundle, the search space has conjunctive (and) structure. For example, the problem

$$\phi \wedge \psi \text{ ?- } \phi \vee \psi,$$

in the search space $Space(LK)$, is the root of bundles arising from $\wedge R$, with two branches, and $\vee R$, with one branch. In the search space $Space(LJ)$, two distinct bundles, each with one branch, arise from $\psi \vee \psi$.

Thus the exploration of a search space requires navigation between disjunctive choices: one might make a choice, such as between the two branches of $Space(LJ)$ generated by the two cases of the $\vee R$ operator, explore that branch of the search space, and perhaps *fail*. One then *backtracks* to the point at which the choice was made, and tries the other branch. Thus backtracking is a key, and we suggest perhaps the prototypical, control mechanism in proof-search. Indeed, the lack of a full permutation theorem for intuitionistic propositional sequent calculus [17, 33, 31], with the consequence that the order of the propositional rules used is critical in the finding of a proof, renders backtracking an essential component of the control of a search for a proof in LJ. To see this, consider the following example, in which the use first of $\supset L$ on $p \supset q$ leaves the

¹⁹Whilst this graphical notation is useful for defining search spaces, it is not convenient for performing specific reductions, for which we revert to the use of “proof trees”.

subsequent development of the left-hand branch of the reduction doomed to failure, even though the endsequent is provable:²⁰

$$\begin{array}{c}
 \text{fails}^{(1)} \quad \frac{\vdots}{p \supset q, r \supset s \text{ ?- } p} \quad \frac{\text{succeeds} \quad \frac{\vdots}{r \supset s, q, r \text{ ?- } s} \quad \frac{\text{succeeds} \quad \frac{\vdots}{r \supset s, q, r, t \text{ ?- } t}}{r \supset s, s \supset t, q, r \text{ ?- } t} \supset L_{s \supset t}}{r \supset s, s \supset t, q \text{ ?- } r \supset t} \supset R \\
 \hline
 p \supset q, r \supset s, s \supset t \text{ ?- } r \supset t \quad (1) \supset L_{p \supset q}
 \end{array}$$

After the first $\supset L$, we can see that the left-hand branch will fail, and we must backtrack to (1) and make a different choice of reduction. We might try $\supset L_{r \supset s}$ instead. Such a control step lies outside the logical structure we have so far established but we can give a logical account of it by considering the intuitionistic calculus LJ to be embedded in the classical sequent calculus, LK. We quickly review the main points from [33] in this context before proceeding to characterize backtracking.

In general, every intuitionistic sequent derivation arises as a subderivation of a classical sequent derivation via (for example) Dummett's presentation of intuitionistic logic as a multiple-conclusion sequent calculus [4]. Because the classical $\supset R$ rule allows multiple succedents in the premiss, two different intuitionistic sequent derivations, which are not identical up to a permutation of inference rules, can be subderivations of the same classical derivation up to a choice of axioms. For example, consider the following two intuitionistic reductions:

$$\frac{\frac{\text{Ax}}{\psi, \phi \text{ ?- } \psi}}{\psi \text{ ?- } \phi \supset \psi, \chi \supset \psi} \supset R \quad \text{and} \quad \frac{\frac{\text{Ax}}{\psi, \chi \text{ ?- } \psi}}{\psi \text{ ?- } \chi \supset \psi, \phi \supset \psi} \supset R.$$

They arise as restrictions to intuitionistic logic of the following classical reduction:

$$\frac{\frac{\text{Ax}}{\psi, \phi, \chi \text{ ?- } \psi, \psi}}{\psi, \chi \text{ ?- } \phi \supset \psi, \psi} \supset R.$$

Similarly, in LK viewed as reductive system, the $\supset L$ rule has the form

$$\frac{\Gamma \text{ ?- } \phi, \Delta \quad \Gamma, \psi \text{ ?- } \Delta}{\Gamma, \phi \supset \psi \text{ ?- } \Delta},$$

in which the Δ is retained in both premisses. Using this operator instead of its intuitionistic counterpart, we are able to restart the computation at (2), and proceed to apply the necessary $\supset R$:

$$\begin{array}{c}
 \text{succeeds} \\
 \vdots \\
 \frac{\vdots}{r \supset s, s \supset t, r \text{ ?- } t, p} \supset L_{s \supset t} \\
 \frac{\vdots}{r \supset s, s \supset t \text{ ?- } \boxed{r \supset t}, p} \supset R \\
 \frac{\vdots}{r \supset s, s \supset t \text{ ?- } p, \boxed{r \supset t}} \supset R \quad (2) \text{ Exchange} \quad \frac{\text{as above} \quad \frac{\vdots}{r \supset s, s \supset t, q \text{ ?- } r \supset t} \supset R}{r \supset s, s \supset t \text{ ?- } r \supset t} \supset R \\
 \hline
 p \supset q, r \supset s, s \supset t \text{ ?- } r \supset t \quad (1) \supset L_{p \supset q}
 \end{array}$$

²⁰We adopt the notation R_ϕ to denote the instance of the operator O generated by the formula ϕ , e.g., $\supset L_{p \supset q}$.

Note, in particular, the use of Exchange at (2). From the point of view of the $\lambda\mu\nu$ -calculus, the necessary $\supset R$ rule is applicable only if the implicational formula is leftmost in the succedent.

A successful classical reduction for a problem $\Gamma \text{ ?- } \phi$ yields a classical proof but not necessarily an intuitionistic proof. So, in order to exploit the structural and combinatorial advantages of classical reduction for intuitionistic logic, we must be able to calculate *syntactically* whether a given classical reduction determines an intuitionistic proof.

To do this we represent the sequent calculus LK in the $\lambda\mu\nu$ -calculus (see [33]). More precisely, we represent LK in the $\lambda\mu\nu\epsilon$ -calculus, *i.e.*, the $\lambda\mu\nu$ -calculus with explicit substitutions. If we represent the classical sequent calculus in the $\lambda\mu\nu\epsilon$ -calculus, then we can calculate whether a successful classical reduction determines the existence of an intuitionistic proof by analyzing the structure of the $\lambda\mu\nu\epsilon$ -term which realizes the classical proof (see [33]).

We repeat here the basic idea. Consider the difference between the $\supset R$ rule in the classical calculus, LK,

$$\frac{\Gamma, \phi \vdash \psi, \Delta}{\Gamma \vdash \phi \supset \psi, \Delta} \supset R,$$

and the form of its restriction to capture intuitionistic implication, as in Dummett's multiple-conclusioned calculus [4],

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \supset \psi, \Delta} \supset R.$$

Here the key point is that a built-in Weakening,²¹ by Δ , is required. To see this, consider the following reduction:

$$\begin{array}{c} \frac{\frac{\frac{\frac{}{Ax}}{\psi, \phi, \theta \text{ ?- } \tau, \psi}}{\psi, \phi, \theta \text{ ?- } \theta \supset \tau, \psi}}{\psi, \phi \text{ ?- } \psi, \theta \supset \tau} \supset R}{\psi \text{ ?- } \phi \supset \psi, \theta \supset \tau} \text{Exchange} \end{array}$$

We need to be able to detect that the use of the $\supset R$ operator to reduce the formula $\theta \supset \tau$ is superfluous, and so conclude that we could have simply deleted $\theta \supset \tau$ at the first $\supset R$ reduction and so conclude that the initial problem, $\psi \text{ ?- } \phi \supset \psi, \theta \supset \tau$ has an intuitionistic proof.

In [33] we developed the the notion of an *intuitionistic term* in the $\lambda\mu\nu\epsilon$ -calculus to solve this problem. We repeat the basic definitions here:

Definition 44 We define Weakening terms and Weakening occurrences of names by induction over the structure of terms as follows:

1. $\mu\alpha.t$ is a Weakening term if all occurrences of α in t are Weakening occurrences;
2. A term t of type \perp is always a Weakening term;
3. $\langle t, s \rangle$ is a Weakening term if t and s are Weakening terms;
4. $\lambda x:\phi.t$ is a Weakening term if t is a Weakening term and if x has only Weakening occurrences in t ;
5. The outermost occurrence of α in $[\alpha]t$ and $\langle \alpha \rangle t$ is a Weakening occurrence if t is a Weakening term;
6. $\nu\alpha.t$ is a Weakening term if t is a Weakening term and all occurrences of α are Weakening occurrences;

²¹The Weakening rules are

$$\frac{\Gamma \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta} WL \quad \text{and} \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \Delta'} WR.$$

7. All occurrences of \perp in t are Weakening occurrences;
8. The occurrence of the variable x in tx is a Weakening occurrence if t is a Weakening term and x is not free in t . In this case, the term tx is a Weakening term as well;
9. $t\{u/x\}$ is a Weakening term if t is a Weakening term.

Definition 45 Call a $\lambda\mu\nu\epsilon$ -term intuitionistic if in any subterm $\lambda x:\phi.t$ which is not a Weakening term, all occurrences of free names are Weakening occurrences.

The idea behind these definitions is that intuitionistic terms identify those implicational realizers, i.e., terms of the form $\Gamma \vdash \lambda x:\phi.t : \phi \supset \psi, \Delta$, in which all of the subterms of $\lambda x:\phi.t$ corresponding to formulæ in Δ arise from Weakenings. In [33] we proved the following Theorem:

Theorem 46 (intuitionistic provability) Let Φ be an LK-proof of $\Gamma \vdash \phi, \Delta$ and let t_Φ be the corresponding $\lambda\mu\nu\epsilon$ -term. Then t_Φ is an intuitionistic term iff $\Gamma \vdash \phi, \Delta$ is intuitionistically provable. \square

If we translate intuitionistic LJ-reductions into classical reductions, backtracking may occur at two points: firstly, at the $\supset L$ rule, and secondly at the $\vee R$ -rule. In both cases we lose potentially useful side formulæ when we apply the reduction operator. This can be captured semantically as follows:

Theorem 47 (backtracking) An intuitionistic reduction contains a possible backtracking point before the reduction operator R if and only if for the translation of the reduction into a classical reduction, the corresponding reduction operator R with sufficient premisses $\Gamma_i \vdash \phi_i, \Delta_i$ and putative conclusion $\Gamma \vdash \phi, \Delta$, there exists a j such that the fibres $\mathcal{E}(1, \llbracket \Delta \rrbracket_\emptyset^1)$ and $\mathcal{E}(1, \llbracket \Delta_j \rrbracket_\emptyset^1)$ are not identical.

Proof All left-operators except $\supset L$ leave the right-hand sides of sequents unchanged, and hence $\mathcal{E}(1, \llbracket \Delta \rrbracket_\emptyset^1)$ and $\mathcal{E}(1, \llbracket \Delta_j \rrbracket_\emptyset^1)$ are identical for all j . These operators also do not give rise to a possible backtracking point. For the operator $\supset L$, $\mathcal{E}(1, \llbracket \Delta \rrbracket_\emptyset^1)$ and $\mathcal{E}(1, \llbracket \Delta_1 \rrbracket_\emptyset^1)$ are not identical, and indeed $\supset L$ gives rise to a backtracking point. All right-operators except $\vee R$ do not modify the side-formulæ on the right-hand side, and hence $\mathcal{E}(1, \llbracket \Delta \rrbracket_\emptyset^1)$ and $\mathcal{E}(1, \llbracket \Delta_j \rrbracket_\emptyset^1)$ are identical for all j . These operators also do not give rise to a possible backtracking point. The $\vee R$ -rule does change the side formulæ on the right-hand side and models the intuitionistic $\vee R$ -rule, which indeed gives rise to a backtracking point. Also, $\mathcal{E}(1, \llbracket \Delta \rrbracket_\emptyset^1)$ and $\mathcal{E}(1, \llbracket \Delta_1 \rrbracket_\emptyset^1)$ are not identical. \square

5 Continuations as a Model for Proof-search

In this section we describe continuations as a model for proof-search. We provide an instance of a classical reduction structure using continuations and analyze backtracking using continuations by specializing the results of the previous section.

In the denotational semantics of programming languages, e.g., [35, 26], in which programs are given a functional interpretation over structures such as the category of *complete partial orders*, an important technique is to interpret not only the *linguistic constructs* of the programming language but also its *control régime*. The semantic structures commonly used for this purpose are called *continuations*.

The idea is that a continuation models a *change of control* during the evaluation of a program with respect to given data: we temporarily suspend the current computation, carry out another, subsidiary, one and after a while resume the original one. Thus a continuation describes how to complete the subsidiary computation and return to the original computation. Continuations are commonly used to describe, *inter alia*, backtracking [35, 10], co-routines [35] and evaluation strategies [25]. A survey of the various origins of the idea can be found in [32].

Rather than attempt a general definition, we describe a category of continuations, introduced by Hofmann and Streicher [12], which can be extended so as to correspond to a semantics of classical proofs as represented by the terms of the $\lambda\mu\nu$ -calculus.

The $\lambda\mu$ -calculus can be used to describe continuations as follows: a continuation of type ϕ is described as the type $\neg\phi$. The intuition is that a continuation expects a term of type ϕ and produces some value which is never used because the control context changes. One could take any type R (for *responses*) for the type of these values, but as it is never used, the $\lambda\mu$ -calculus uses \perp for the type of these values. The creation of a continuation is then described by a term of type $\phi \supset \neg\neg\phi$ because it transforms a value of type ϕ into a continuation $\neg\phi$. The other direction, namely the evaluation of a continuation, gives a term of type $\neg\neg\phi \supset \phi$. With these two control operators it is possible to define an operational semantics which treats each term as a continuation rather than having a value.

This syntactic view has a semantic counterpart: Hofmann and Streicher define a category of continuations as a category C with a distinguished class T of objects of C called type objects and a distinguished type object R of responses. In addition there is a chosen Cartesian product $\Gamma \cdot \phi$ for every object Γ and type ϕ , and chosen terminal objects $[]$ and $1 \in T$. Moreover, for each type object ϕ there is a chosen exponential $R^\phi \in T$, and for any two type objects ϕ and ψ a chosen Cartesian product $R^\phi \cdot \psi \in T$ of R^ϕ and ψ . A $\lambda\mu$ -term $\Gamma \vdash t: \psi, \Delta$ is interpreted in such a category as a map $R^{[\Gamma]} \cdot [\Delta] \rightarrow R^{[\psi]}$.

To interpret conjunctions, we ask in addition for sums of types in the category, and can then define $[\phi \wedge \psi] = [\phi] + [\psi]$, and use standard isomorphisms involving sums, products and exponentials to define the interpretation of $\lambda\mu$ -terms involving products or projections.

The classical disjunction requires the closure of T under products $\phi \cdot \psi$ for every $\phi, \psi \in T$: we can define

$$[\phi \vee \psi] = [\phi] \cdot [\psi]$$

and use the natural isomorphism between

$$\text{hom}(R^{[\Gamma]} \cdot [\Delta], R^{[\phi] \cdot [\psi]}) \quad \text{and} \quad \text{hom}(R^{[\Gamma]} \cdot [\Delta] \cdot [B], R^{[\phi]})$$

as the categorical counterpart of the introduction and elimination rules for disjunction.

A similar construction for the intuitionistic disjunction \oplus seems to be more difficult to obtain. For the soundness theorem we require

$$\text{hom}(R^{[\phi \oplus \psi]} \cdot [\Delta], R^{[\chi]}) \cong \text{hom}(R^{[\phi]} \times [\Delta], R^{[\chi]}) \cdot \text{hom}(R^{[\psi]} \cdot [\Delta], R^{[\chi]})$$

but there is no obvious way of defining $[\phi \oplus \psi]$ in a Cartesian closed category such that $R^{[\phi \oplus \psi]} \cong R^{[\phi]} + R^{[\psi]}$. We show in [27] that intuitionistic and classical disjunction do not coincide proof-theoretically: a $\lambda\mu\nu$ -calculus in which classical and intuitionistic disjunction coincide is trivial in the sense that all terms of the same type are equal.

Hofmann and Streicher prove completeness for $\lambda\mu$ -categories by defining a continuation category C from the syntax of the $\lambda\mu$ -calculus. Objects are (continuation) contexts $\Delta = \phi_1^{\alpha_1}, \dots, \phi_m^{\alpha_m}$; a morphism from Δ to ϕ is a certain $\lambda\mu$ -term t such that $\vdash t: \phi \supset \perp, \Delta$. The intuition is that t transforms the name α^i of type ϕ_i to a continuation of type ϕ , which is the type $\phi \supset \perp$. The condition on the term is that for any observer o (any $\lambda\mu$ -term of type $\neg\neg\phi$) the two possible terms for execution of the continuations t by the observer, namely ot and $t(\mu\alpha^\phi.o(\lambda x: \phi. [\alpha]x))$, are equal. The type of responses is fixed as $\perp \supset \perp$. It follows from the naturality of their definitions, *i.e.*, they respect substitution, that the completeness result can be extended to cover conjunction and classical disjunction.

Hofmann and Streicher also prove that the continuation categories are universal for the $\lambda\mu$ -calculus in the sense that for each $\lambda\mu$ -theory (*i.e.*, a $\lambda\mu$ -calculus with some additional judgemental equalities between terms) there is a continuation category (namely the term model) such that there

is a map from this model to any other $\lambda\mu$ -model which respects the interpretation of $\lambda\mu$ -terms in both models. Again, it follows from the naturality of their definitions, *i.e.*, they respect substitution, that the universality result can be extended to cover conjunction and classical disjunction.

The completeness of our categorical model implies that we must be able to transform each continuation category into a $\lambda\mu\nu$ -structure. For this construction, we view this category as a category of display maps [14]; then we exploit a standard construction which transforms categories of display maps into fibrations [15]. We sketch this construction, but omit the detailed verification that the structure we define and which we call a *continuation fibration* is indeed a $\lambda\mu\nu$ -structure, as follows:

- The base category \mathcal{B} has as objects the objects of \mathcal{C} and all morphisms necessary to make \mathcal{C} a Cartesian category;
- Objects of the fibre \mathcal{F}_Δ are projection morphisms $\Delta \cdot \phi \rightarrow \Delta$;
- Morphisms from $\Delta \cdot \phi \rightarrow \Delta$ to $\Delta \cdot \psi \rightarrow \Delta$ are morphisms f in \mathcal{C} such that $\pi_\psi \circ f = \pi_\phi$, where π_ψ and π_ϕ are the projections corresponding to $\Delta \cdot \psi$ and $\Delta \cdot \phi$, respectively;
- Given a morphism $f: \Gamma \rightarrow \Delta$ the functor $\mathcal{F}(f)$ transforms an object $\Delta \cdot A \rightarrow \Delta$ to $\Gamma \cdot \phi \rightarrow \Gamma$ and a morphism h into $\pi' \circ (\text{Id} \times h) \circ (\text{Id} \times f)$, where π' is the projection from $\Gamma \cdot \Delta \cdot \psi$ to $\Gamma \cdot \psi$;
- The object \perp is R ;
- The isomorphism between $\mathcal{F}_\Delta(\chi, \phi)$ and $\mathcal{F}_{\Delta \cdot \phi}(\chi, \perp)$ is captured by the bijection between $\text{hom}(\Delta \cdot \phi, R)$ and $\text{hom}(\Delta, R^\phi)$ in \mathcal{C} ;
- The naturality and Beck-Chevalley condition of the bijection ζ follow from the fact that $\mathcal{F}(f)$ is defined by composition.

The verification that interpretations of $\lambda\mu\nu$ are indeed well-defined in this structure, so yielding our definition of a $\lambda\mu\nu$ -model, is routine.

Finally, we remark that Hofmann and Streicher also show that the interpretation of a $\lambda\mu$ -term t in the syntactic continuation category is obtained by replacing each object variable x by a term which describes the execution of a continuation given by a new name α . This interpretation transforms each term into a continuation. This property too extends to $\lambda\mu\nu$.

Next, we describe how to construct a classical reduction structure out of continuations.

Definition 48 *Let \mathcal{C} be any continuation category \mathcal{C} . A continuation reduction structure (\mathcal{E}, F) is given by*

- *a strict indexed category $\mathcal{E}: (\mathcal{B} \cdot \mathcal{D})^{\text{op}} \rightarrow \mathbf{Cat}$, where the category \mathcal{B} is the category \mathcal{C} again, and the category \mathcal{D} is the free cartesian category over the objects of \mathcal{C} and the functor \mathcal{E} is defined by $\mathcal{E}(A, B) = \mathcal{F}(A, \cdot B)$;*
- *a category \mathcal{W} worlds, which is the full subcategory of \mathcal{C} of objects R^A ;*
- *a functor F , which is the inclusion functor from \mathcal{W} into \mathcal{C} ;*

It is easy to see that every continuation reduction category is indeed a classical reduction structure:

Theorem 49 *For each continuation category \mathcal{C} , the continuation reduction structure (\mathcal{E}, F) is a classical reduction structure.*

Proof Routine verification, using the fact that the functor F is an inclusion. □

Now we show how to interpret backtracking in this setting. In an earlier section, we related backtracking to the change of fibre in an classical reduction structure. If we specialise this to the classical reduction structure built from continuations, it turns out that backtracking is captured by suspending the current continuation and selecting another continuation. Technically, this is given by the isomorphism

$$\frac{\frac{\text{Hom}(R^\Gamma \cdot \psi, R^\phi)}{\text{Hom}(R^\Gamma \cdot \phi \cdot \psi, R)}}{\text{Hom}(R^\Gamma \cdot \phi, R^\psi)}$$

which is modelled by a change of fibre in the classical reduction structures. The precise theorem is as follows:

Theorem 50 *A reduction operator R gives rise to a backtracking point if its translation into a continuation reduction structure applies a change of continuation to the sufficient premisses.*

Proof Direct consequence of Theorem 47. □

6 A Games Semantics for Proof-search

We conclude with an example of our semantics — of intuitionistic reduction with backtracking, embedded in classical reduction — which corresponds closely to our intuitions about the nature of constructing proofs: *i.e.*, a games semantics for proof-search.

This games semantics is an extension of the games semantics we described in § 2.2 to games for classical logic. The main difference between the games for intuitionistic logic and those for classical logic is a consequence of the fact that for classical logic we are working with sequents with multiple conclusions, $\Gamma \vdash \Delta$, with the intuitive meaning that (at least) one of the formulæ in Δ must to be proved, whereas in intuitionistic logic we work with only one conclusion. This means that, in classical games, when Opponent challenges a formula ϕ in Δ , Proponent might choose to defend a different formula ψ in Δ , which has to be accepted also as a valid defense of ϕ .

The definitions of arenas, moves and justification for classical games are the same as those for intuitionistic games. We call a strategy (play) classical if it is the one for classical games. Otherwise we call the strategy (play) intuitionistic.

The conditions for classical plays are not as strong as the conditions for intuitionistic plays. In particular, the rules for disjunction have been changed to allow Proponent to select both disjuncts, thereby possibly violating the disjunction property of intuitionistic logic. More precisely, we have relaxed Clause (vi) and Clause (vii). We drop the latter clause, and replace the former as follows:

Definition 51 *A play for an arena \mathcal{A} is a sequence of moves m_1, \dots, m_n such that conditions (i) – (v) for intuitionistic plays, and the following additional condition are satisfied:*

- (vi) *For any P -answer m_i there exists a O -question m_k and an O -answer m_j such that m_i is hereditarily justified by m_k , m_j is an O -answer with the same label as m_k or \perp and $k < j < i$ and that the nodes corresponding to m_k and m_j in the arena are on a path which does not contain a P -node n labelled \supset such that the nodes corresponding to m_i and m_j are its children or identical to it;*

This relaxation captures the possibility of pending O -questions (arising from the multiple conclusions on the right-hand side) being answered as well as the immediate justifying question.

Compared to a games semantics for natural deduction, we allow both Opponent and Proponent more freedom: both players can make several moves at a time, which are subject to fewer restrictions. In this way, we capture the possibility of applying reduction operators to several sequents independently. We also capture the possibility of sequences of blocks of left and right rules in a play.²²

²²This latter possibility is critical for modelling proof procedures such as resolution.

This games semantics is sound for classical logic:

Theorem 52 *For any formula ϕ and classical strategy Φ for ϕ there exists a classical sequent calculus proof of ϕ .*

The proof follows the same line as the proof for the corresponding theorem for intuitionistic games (Theorem 25). Again, we have to show a stronger version of the theorem with a stronger notion of strategy.

Definition 53 *For a set A , of propositional atoms or \perp , and a sequence ϕ_1, \dots, ϕ_k of formulæ, define an A, ϕ_1, \dots, ϕ_k -strategy for the formula ϕ to be any strategy for ϕ where both players may make additional moves according to the arenas for ϕ_1, \dots, ϕ_k .*

The key lemma is now the following:

Lemma 54 *Given formulæ ϕ_1, \dots, ϕ_k and any set A of O -answers with labels p_1, \dots, p_n and a A, ϕ_1, \dots, ϕ_k -strategy for a formula ϕ there is a classical proof of $p_1, p_n \vdash \phi, \phi_1, \dots, \phi_k$.*

Proof By induction over the structure of $\phi, \phi_1, \dots, \phi_k$. As the definition of A, ϕ_1, \dots, ϕ_k -strategy is invariant under permutation of any of the ϕ_i 's and ϕ and sequent calculus admits the exchange rule, it suffices to do a case analysis regarding the structure of ϕ . We will write Δ for the sequence of formulæ ϕ_1, \dots, ϕ_k and Γ for the sequence p_1, \dots, p_n .

Atoms: Firstly, assume $\phi = p$ for some propositional atom p , and $\phi_1, \dots, \phi_k = q_1, \dots, q_k$, where all q_i 's are atoms or \perp . Any possible strategy starts by Opponent asking at least one question labelled p or q_1, \dots, q_k . Proponent only has an answer if either $p_i = p$, for some i , or $p_i = q_j$, for some i and j . In both cases, the classical axiom $p_1, \dots, p_n \vdash p, q_1, \dots, q_k$ is the desired sequent calculus proof;

$\psi_1 \vee \psi_2$: Any possible strategy starts with Opponent asking question corresponding to the root of the arena for $\psi_1 \vee \psi_2$. There are now several cases. If Opponent never asks any initial question for the arenas ψ_1 and ψ_2 , then the given strategy is also a strategy for ϕ_1, \dots, ϕ_k . Hence there is a sequent calculus proof of $\Gamma \vdash \Delta$ and hence also of $\Gamma \vdash \psi_1 \vee \psi_2, \Delta$. If Proponent never asks the question corresponding to the node labelled R (L) of this disjunction or Opponent never asks any of the initial questions of the arena for ψ_2 (ψ_1) then the given strategy is also a strategy for ψ_1 (ψ_2). By induction hypothesis there is a sequent calculus proof of $\Gamma \vdash \psi_1, \Delta$ ($\Gamma \vdash \psi_2, \Delta$) and hence also a sequent calculus proof of $\Gamma \vdash \psi_1 \vee \psi_2, \Delta$. If Opponent asks any initial questions for both arenas ψ_1 and ψ_2 , then the strategy has to consider all initial moves for ψ_1 and ψ_2 . Hence by induction hypothesis for ψ_1, ψ_2, Δ there exists a sequent calculus proof $\Gamma \vdash \psi_1, \psi_2, \Delta$ and hence also a sequent calculus proof of $\Gamma \vdash \psi_1 \vee \psi_2, \Delta$;

$\psi_1 \wedge \psi_2$: Because every question and answer of a strategy for ψ_1 and ψ_2 has to be justified eventually by an initial move for ψ_1 and ψ_2 it is possible to obtain one strategy for ψ_1 and one strategy for ψ_2 from the given strategy. Hence by induction hypothesis we obtain sequent calculus proofs for $\Gamma \vdash \psi_1, \Delta$ and $\Gamma \vdash \psi_2, \Delta$. Hence one obtains also a sequent calculus proof for $\Gamma \vdash \psi_1 \wedge \psi_2, \Delta$;

$\phi' \supset \psi$: There are several subcases. Firstly, assume $\phi' = \psi_1 \wedge \psi_2$. Then $(\psi_1 \wedge \psi_2) \supset \psi$ is equivalent to $\phi_1 \supset \psi_2 \supset \psi$, and the arenas for $(\psi_1 \wedge \psi_2) \supset \psi$ and $\psi_1 \supset \psi_2 \supset \psi$ are identical. Hence we consider the case $\psi_1 \supset \psi_2 \supset \psi$ instead.

Secondly, assume $\phi' = \sigma \vee \tau$. Now define two A, Δ -strategies Φ_1 and Φ_2 for $\sigma \supset \psi$ and $\tau \supset \psi$ respectively, where the moves of both players in Φ_1 and Φ_2 are the moves of Φ which are justified by moves not hereditarily justified by τ or σ respectively. By the induction hypothesis, we obtain sequent calculus proofs for

$$\Gamma \vdash (\sigma \supset \psi), \Delta \quad \text{and} \quad \Gamma \vdash \tau \supset \psi, \Delta.$$

Hence there is also a sequent calculus proof for $\Gamma \vdash (\sigma \vee \tau) \supset \psi, \Delta$.

Thirdly, suppose $\phi' = \sigma \supset \tau$. Again, define A, Δ -strategies Φ_1 for $\tau \supset \psi$ and Φ_2 for σ where the moves of both players are the ones not hereditarily justified by σ or τ respectively. By induction hypothesis we obtain sequent calculus proofs for $\Gamma \vdash \tau \supset \psi, \Delta$ and $\Gamma \vdash \sigma, \Delta$. Hence there is also a sequent calculus proof for $\Gamma \vdash (\sigma \supset \tau) \supset \psi, \Delta$.

Fourthly, suppose ϕ' is an atom p . Again, there are two cases. Consider a A, Δ -strategy for $p \supset \psi$ without a Proponent-question corresponding to p . In this case, the A, Δ -strategy for $p \supset \psi$ is in fact a strategy for ψ , and by induction hypothesis there is a sequent calculus proof of $\Gamma \vdash \psi, \Delta$, hence also a proof of $\Gamma \vdash p \supset \psi, \Delta$. Now suppose there is a Proponent-question corresponding to p . In this case the strategy which removes the Proponent-question and O-answer for p is a $A \cup \{p\}, \Delta$ -strategy for ψ . By induction hypothesis there is a sequent calculus proof for $\Gamma, p \vdash \psi, \Delta$ and hence also for $\Gamma \vdash p \supset \psi, \Delta$.

Finally, suppose $\phi' = \perp$. In this case there is always a sequent calculus proof of $\Gamma, \perp \vdash \psi, \Delta$, and hence also a proof of $\Gamma \vdash \perp \supset \psi, \Delta$.

□

The games semantics Ong presents in [23] for the $\lambda\mu$ -calculus (without disjunction) uses *scratchpads* to model classical logic. Scratchpads are separate plays to be started by Proponent whenever he chooses. As we consider disjunction as well, we have extended the definition of an arena and introduced the concept of switching moves (the moves labelled L and R) to model the $\lambda\mu\nu$ -calculus. Proponent choosing a move labelled R corresponds to the switch of fibres in the $\lambda\mu\nu$ -structures, which is captured by changing to a scratchpad in Ong's model.

Next we describe the additional structure we need to model reductions and searches. The additional structure is very similar to the one we already introduced for the case of games for intuitionistic reductions and searches.

To formulate this extension we need a lemma about substitutions. If one substitutes arbitrary formulæ for propositional variables in a proof, one still obtains a valid proof. This substitution lemma has an important analogon for games:

Lemma 55 *Suppose we have a strategy for the arena of a type ϕ which contains a propositional variable A . Then there also a strategy for the arena of type $\phi[\psi/A]$, where ψ is any formula.*

Proof We only sketch the proof here. By definition of plays, in all plays defined by the strategies Opponent asks a question labelled A before Proponent does, and Opponent's answer is then used by Proponent to answer Opponent's original question. Hence Proponent can use a copycat-strategy whenever the opponent makes a move in the arena for ψ . □

To model reductions, we use *oracles*, i.e., additional plays which Proponent may start at will.

Definition 56 *A strategy with oracle of type ϕ is a strategy where in addition Proponent is allowed to play using an additional arena for ϕ .*

The instantiation of non-axiom leaves of a reduction with reductions is modelled by the substitution of strategies for oracles.

Definition 57 *Suppose Ψ is a strategy with oracle of type ϕ and Φ is a strategy of type ϕ . We define the substitution of Φ for the oracle in Ψ to be the strategy Ψ except that we replace every answer which is a move given by the arena for Φ by the move obtained by using Φ to answer Ψ 's move in ϕ , then using Ψ to answer this move and so on until Ψ answers with a move outside the arena for ϕ .*

Before we can construct a classical reduction structure from games, we need some preliminary notation.

Definition 58 Suppose \mathcal{C} is the free Cartesian category over the set of formulæ and assume π is a morphism from (ϕ_1, \dots, ϕ_n) to (ψ_1, \dots, ψ_m) and assume that Φ is a strategy for $\psi \vee \psi_1 \vee \dots \vee \psi_m$. Furthermore, let $B_{i,1}, \dots, B_{i,k_i}$ be the arenas of ψ_i and $A_{j,1}, \dots, A_{j,l_j}$ be the arenas of ϕ_j , and let A_1, \dots, A_n be the arenas of ϕ .

We define the strategy $\pi^*(\Phi)$ to be the strategy for $\psi \vee \phi_1 \vee \dots \vee \phi_n$ answering any question in the arena for ψ by the answer Φ would give to the corresponding question, and by answering any Opponent move in the part of the arena selecting a subarena for ψ_j by the Proponent move selecting the corresponding subarena for ϕ_i , where π maps ϕ_i to ψ_j , and answering any move in any subarena ψ_j by the answer Φ gives to the corresponding subarena in ϕ_i .

We now describe how to construct a classical reduction structure from this notion of game. Intuitively, the base category \mathcal{B} of a reduction structure models the collection of indeterminates. A reduction with indeterminates is modelled as a *game with oracles*. Hence the category \mathcal{B} consists of formulæ as objects (these represent the available oracles) and of games with oracles as morphisms. The indexing functor models substitution of games for oracles. As the category of worlds, we take compositions of reduction operators, as in the construction of the term models in § 3.

The precise definition of the classical reduction structure obtained from games is given in the proof of the following proposition:

Proposition 59 *Games form a classical reduction structure.*

Proof We present here only the definition of the categories involved; the natural transformations are straightforward.

The category \mathcal{C} is the free Cartesian category over the set of formulæ.

The category \mathcal{B} has as objects finite lists of formulæ (ϕ_1, \dots, ϕ_n) and as morphisms from (ϕ_1, \dots, ϕ_n) to $(\phi'_1, \dots, \phi'_m)$ finite lists Φ_1, \dots, Φ_m of strategies such that Φ_i is a strategy for ϕ'_i possibly with oracles of type ϕ_1, \dots, ϕ_n . We define composition of two morphisms

$$(\Phi_1, \dots, \Phi_n): (\sigma_1, \dots, \sigma_k) \rightarrow (\phi_1, \dots, \phi_n) \quad \text{and} \quad (\Psi_1, \dots, \Psi_m): (\phi_1, \dots, \phi_n) \rightarrow (\psi_1, \dots, \psi_m)$$

in \mathcal{B} as the list of strategies $(\psi'_1, \dots, \psi'_m)$, where ψ'_i is the strategy ψ_i with every answer which arises from the arena for ϕ_j is replaced by the move obtained by first using the strategy Φ_j to answer this move, then Ψ to answer this move and so on until ψ_i answers with a move outside the arena for ϕ_j .

For each pair of finite lists of formulæ, (ϕ_1, \dots, ϕ_n) and (ψ_1, \dots, ψ_m) , we define a category $\mathcal{F}((\phi_1, \dots, \phi_n), (\psi_1, \dots, \psi_m))$, where the objects are formulæ and the morphisms from ϕ to ψ strategies for $\phi \supset (\psi \vee \phi_1 \vee \dots \vee \phi_n)$, with oracles of type ψ_1, \dots, ψ_m . We define composition in the category

$$\mathcal{F}((\phi_1, \dots, \phi_n), (\psi_1, \dots, \psi_m))$$

in the same way as in the category \mathcal{B} .

For a morphism (Φ_1, \dots, Φ_n) in \mathcal{B} and π in \mathcal{C} we define a functor

$$\mathcal{E}((\Phi_1, \dots, \Phi_n), \pi)$$

by leaving the objects unchanged and assigning to each strategy Φ the strategy $\pi^*(\Phi')$, where Φ'_i is the strategy obtained by substituting Φ_i for the indeterminate of type ϕ_i in Φ .

As category of worlds, we take the free Cartesian category generated from ground objects W_R , where R is an LK-reduction operator, and ground morphism $a_R: 1 \rightarrow W_R$ for each reduction operator R . The functor F is defined as the functor assigning to W_R the object

$$(\Gamma_1 \supset \phi_1 \vee \Delta_1) \wedge \dots \wedge (\Gamma_n \supset \phi_n \vee \Delta_n) \supset (\Gamma \supset \phi \vee \Delta)$$

where R is a reduction operator with sufficient premisses $\Gamma_i \vdash \phi_i, \Delta_i$ and putative conclusion $\Gamma \vdash \phi, \Delta$, and to the morphism a_R the canonical derivation given by R . \square

Note that this highly intensional category is non-trivial: equality between morphisms is essentially equality between partial functions. As the arenas for \perp and \top and for ϕ and $\neg\neg\phi$ are different, strategies for them cannot be equal. If we were to try to define an extensional collapse of this category, we should have to be careful to ensure that the arenas for $\neg\neg\phi$ and ϕ be not identified under the collapse.

Now we explain how backtracking is modelled in our game semantics. Backtracking points are captured by the possibility of Proponent making disjunctive choices which are not available when the moves are restricted to intuitionistic games. This is the case when Proponent plays both switching moves and when Proponent plays a P-question m corresponding to a node arising from a $\supset L$ -operator. In the first case, playing the other switching move is not allowed in games for LJ, and in the second case no previously pending O-question can be used to justify the P-answer to the O-question which is the immediate successor to the P-question m .

Backtracking actually occurs when Proponent plays a different switching move, or actually answers a question with a different label using Clause (vi) of the definition of a play.

To illustrate this point, consider an example of the previous section, namely the reduction for the sequent

$$((p \supset q) \wedge (r \supset s) \wedge (s \supset t) \wedge r) \rightarrow t.$$

The arena is given in Figure 6. Then the following play corresponds to the second reduction (with

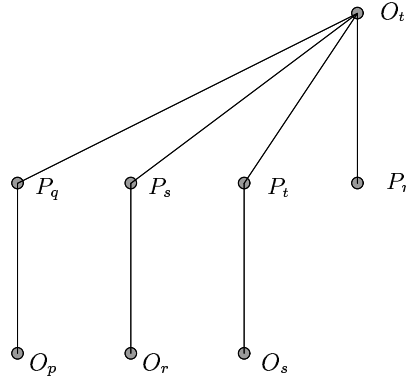


Figure 6: Arena for $(p \supset q) \supset (r \supset s) \supset (s \supset t) \supset r \supset t$

the Exchange) in the § 4:

$$O_t^Q P_q^Q O_r^Q P_t^Q O_s^Q P_s^Q O_r^Q P_r^Q O_r^A P_r^A O_s^A P_s^A O_t^A P_r^A O_q^A P_t^A.$$

where moves by Opponent (Proponent) are denoted by the letter O (P) with subscripts and superscripts, and the subscript indicates the label of the move and the superscript indicates whether the move is a question or an answer.

Note first the contraction involved in this play: the move P_t^Q models both instances of the $\supset L$ -operator reducing $s \supset t$. The backtracking points are the P-questions labelled q , s and t , and backtracking is reached with the move P_r^A : this move is possible only in games for multiple-conclusioned LK, and models the exchange which is necessary to make the reduction succeed.

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